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THE MATHEMATICS TEACHER

Volume XLIV Number 3

What About the Structure of the Mathematics Curriculum?[1]

By F. Lynwood Wren

George Peabody College for Teachers, Nashville, Tennessee

Slightly more than three centuries ago the Latin Grammar School, the first secondary school in the United States, was established in Boston, Massachusetts. This school was organized primarily to prepare boys in Latin grammar and literature for entrance into college. Although mathematics found no place in this curriculum it later gained entrance under the guise of arithmetic and through the avenue of the writing schools. Once admitted, arithmetic continued to gain significance as an integral part of the curriculum.

From this one school with a few boys as pupils and one purpose in its curriculum the program of public secondary education has grown to include approximately 29,000 schools and 5,600,000 pupils of both sexes, not to mention the private schools with approximately one-half million boys and girls. Furthermore, in many quarters today the concept of secondary education has expanded to include the first two years of the college program. Concomitant with this great increase in size in the program of secondary education there has been a vast expansion of purpose. Modern educational problems are rooted in the context of mass education which seems to nurture extensive, and at times seemingly excessive, curriculum design. These problems will always remain complex; there is no way to simplify them. This complexity becomes even more aggravated when we extend our considerations to include the approximately 18,000,000 boys and girls enrolled in the public elementary schools of our country, and is only bounded in expanse when we restrict our thoughts to the contributions of one subject-matter field, such as mathematics, to a functional educational program. The only hope for an intelligent approach to the solution of these problems is through the co-operative effort of well-organized groups who are interested in the formulation and promotion of that program in our schools which will be most significant when evaluated in terms of the educational needs of a school population that is so highly heterogeneous as to interests and abilities.

In the early frontier days the need for mathematics and the motivation for its development were very slender. The

[1] An address delivered before the National Council of Teachers of Mathematics at its Ninth Summer Meeting at Denver, September 1, 1949.

farmer and the ordinary worker needed little beyond the ability to add and subtract. Those boys who aspired to enter some form of trade experience needed to know something of simple computation along with a slight knowledge of common measure, a few very simple fractions, and the use of English and other European monies. The sea-faring man needed to know the basic principles of navigation; the clergyman found astronomy necessary in fixing the dates of religious festivals; and the public official, when fixing territorial boundaries, found surest recourse in surveying. From this meager source flowed the shallow stream of mathematics in sixteenth-century America. Consequently, it is not surprising that mathematics received little attention in the early elementary schools. They were established for the primary purpose of teaching "writing and reading" with an occasional reference to "ciphering." In fact, one who could qualify as an "arithmeticker" was likely to be considered as especially endowed, although this sobriquet implied no more than that the individual was able to perform the simplest of computations.

The low degree of intellectual effort in these early days caused many of the leading statesmen of the period a great deal of concern. Some made distinct efforts to analyze the situation and to get at the causes. The early settlers were not interested in the works in literature and science produced by the masters of Europe. There was even opposition to public education "on the ground that it made boys lazy and dissatisfied with farm life, and led to religious skepticism."

It was not until the latter part of the nineteenth century that real progress began to take place in the development of mathematics in America. By this time there was a fair-sized group of native talent working in the field of mathematics, and there were major influences shaping up to promote significant development in mathematical research. As most important among these influences we might list: the vision of the presidents of certain outstanding universities and their appreciation of the need for promotion of interest in mathematics in this country; the founding of the American Mathematical Society; and a closer contact with European scholars. Many of the young men of that era, who were interested in mathematics, pursued advanced study and sought advanced degrees in mathematics, in European universities, in particular, those of Berlin, Göttingen, and Leipzig. Furthermore, many European mathematicians were imported to this country to teach in our universities, some temporarily and others on a permanent basis.

It was well that the leaders in the field of mathematics should have become so concerned with the promotion of mathematical research in this country. The research worker is the producer of mathematics. It is he who must stay out on the frontier in the expansion of the effectiveness, usefulness, and significance of the field of mathematics as an important aspect of our culture and social structure. The investments of effort and planning made by these early leaders have paid excellent dividends to the stockholders in American culture, for today the American corps of mathematicians occupy the vanguard of mathematical thought.

Thus, we see that during the course of approximately four centuries, vision and scholarly effort have produced a strong and impressive superstructure of enlightened research in the field of mathematics in our country. But what of the understructure of effective teaching and intelligent subject-matter planning at the elementary and secondary levels? Recently, in the city in which I live, crumbling cement was found on the supports of two important bridges across the Cumberland River. Although the superstructures of these bridges were still as strong and impressive as they had ever been, the city engineer immediately restricted travel

on both bridges and called in experts to help determine the full seriousness of the situation. If these experts are deserving of their titles, they will not be over-awed by the impressiveness of the superstructure, but they will be considerably concerned with the effectiveness of the understructure of each bridge. So it is in the field of mathematics. The events of recent years have created an international feeling of awe and admiration for the marvelous accomplishments of mathematical research, at the same time there have been many pointed expressions of concern over faulty teaching and deficient curricula in mathematics at the elementary and secondary levels. Who is the engineer to take charge, and who are the experts to be called in to determine the validity and seriousness of such charges? What should be done?

In 1903 the Central Association of Science and Mathematics Teachers was organized. One of its expressed purposes for organization was to promote better teaching of mathematics and the sciences. In 1915 the Mathematical Association of America was organized for the purpose of contributing to "the advancement of mathematical science by the stimulating of teaching." In 1920 the National Council of Teachers of Mathematics was organized to promote interest in mathematics, especially in the elementary and secondary fields. These three organizations have been instrumental in promoting constructive thought concerning better instruction in mathematics at the elementary, secondary, and collegiate levels of instruction. Furthermore, there are certain state groups and regional groups that have done some excellent work. The fact still remains that there is much to be done.

The history of mathematical development, with its important emphasis on research, prepares us to think at the college and university levels in the perspective of traditional topics and courses and a traditional way of teaching. Is it not likely that we have allowed such thoughts to exercise too strong an influence at the lower levels of instruction? The higher up the educational ladder we climb, the more selective becomes the educable clientele. In fact, at the university level it is desirable that our criteria of educability be shaped after the pattern of high selectivity. Not so at the lower levels! The American pattern of a democratic form of education for all American youth creates problems cut from cloth entirely different from that from which those at the university level are fashioned. At the university level the criteria for the adaptability of student and subject matter might well be summed up in one question. This question takes the form of a single criterion which emphasizes student adroitness. It is: "Is the student sufficiently mature in aptitude, interest, and preparation to enter into the study of a particular body of mathematical subject matter?" At the elementary level such criteria shape themselves into a somewhat converse question which takes the form of a single criterion emphasizing subject-matter adaptability. It is: "Is the subject matter adaptable in selection, organization, and method of presentation to the mathematical immaturity of the student?" In planning the content and teaching procedures of the mathematical program in our schools there should be a gradual transition from the emphasis and implications of the second question to those of the first. Throughout the elementary school and the junior high school the predominating emphasis should be on subject-matter adaptability. Beginning with the senior high school and extending probably through the student's first course in calculus there should be a balance of emphasis between subject-matter adaptability and student adroitness. In the subsequent years of college and university study the controlling criterion should be that of student maturity and adroitness.

The problems of the organization and presentation of mathematics at the ele-

mentary and secondary levels are closely interwoven with problems related to the training of teachers of mathematics. There does exist the problem of what should be done about the program of mathematics in the elementary and secondary schools before we can begin to formulate an intelligent program for the training of teachers of mathematics in these schools. A recent exchange of correspondence with an outstanding teacher of mathematics in one of the larger high schools of the middle west concerning some of the basic problems related to teacher training brought forth the following question from this teacher: "What kind of mathematics is a high school teacher supposed to teach?" This is a very pertinent question and might just as well have been: "What kind of mathematics is a high school teacher or an elementary school teacher supposed to teach?" A student in one of my classes recently posed this difficult question: "What are we supposed to do in the, Seventh Grade when we get pupils that do not know even the basic operational facts of arithmetic?" Is it stretching the imagination too far to suppose that, instead, this student might have asked these two questions: "What mathematical ability is a student supposed to have when he enters the Seventh Grade?" "Is there any real basis for an authoritative answer to this question?" Of course, we can refer the student to the 1923 report of the National Committee on Mathematical Requirements, or to the 1940 report of the Joint Commission. Each of these reports carry statements concerning the "mathematical equipment of the American pupil who has satisfactorily completed the work of the sixth grade." Furthermore, the Second Report of the Commission on Post War Plans, published in 1945, makes a similar declaration. These reports merit praise for the good influence they have exercised and continue to exercise in the interest of improved instruction in mathematics in our schools. Would it not be well, however, to subject to critical review their assumptions as to mathematical attainment and their bases for selection and organization of subject matter? Are they being validated through the test of classroom experience or is there evidence that the mathematical program in our schools is overcast by the shadow of traditional choice, grade-placement, and presentation of subject matter?

In its Second Report the Commission on Post War Plans presents a Check List which it uses to define functional competence in mathematics. This list contains twenty-eight items, and was later expanded in the Guidance Pamphlet to include twenty-nine. The Commission then makes this statement: "Although many of these ideas should be taught in earlier grades, this list of essentials for functional competence in mathematics should constitute the main goals of the work in grades 7 and 8. It is not to be inferred that all the essentials needed for functional competence in mathematics should be taught in grades seven and eight, nor that very many of these ideas can be completely taught to all pupils enrolled in these grades. However, grades 7 and 8 provide a golden opportunity for the mathematics teacher." Are these claims too ambitious? Can they be justified?

Who is to answer all these questions? Does not the principal responsibility for taking the lead in this great task lie in the hands of the Mathematical Association of America and the National Council of Teachers of Mathematics? These two organizations are national in nature and have declared openly in their articles of organization that they are interested in improved instruction in mathematics. The determination of intelligent answers to these questions, and many others that can and should be raised, is a herculean task that will require time, money, and effort. It will demand the maximum of co-operation on the part of these two organizations and should receive the sympathetic interest and active support

of all other groups with a vested interest in mathematics. However, the great need for co-operation does not stop here. While these groups must take the lead, they must seek counsel from the benefits of co-operative thinking with duly organized educational groups whose responsibility it is to study the entire school program from the points of view of curriculum organization and instructional techniques in the perspective of the modern demands of mass education. Some progress has been made in this direction but, in view of what might be achieved, the co-operation of the groups that should be jointly concerned over these problems is very much like the three men who met on the street corner one day. Tom and Sam had been engaged in conversation for some time when Bob approached. Bob recognized Tom and called: "Hi, Tom! How're you getting along?" Tom replied: "Fine! How've you been?" He then motioned for Bob to join him and Sam. As Bob came up Tom shook his hand and asked him this question: "Bob, do you know Sam?" To which Bob replied: "Well, we've howdied but we ain't shook." The above-mentioned organizations have only "howdied" in their efforts to co-operate in the solution of problems related to the educational program in our schools. Why can they not "shake" and get together around a table to discuss these important problems, which should be considered as common problems, and seek intelligent answers which would point toward the shaping of a program of mathematics for our schools which all groups would be willing to accept as authoritative and willing to support actively and constructively? Maybe the Check List proposed by the Commission on Post War Plans might serve as a starting point for the clarifying of issues. Is the amount of time that can be allowed mathematics in the crowded curriculum sufficient to reach the standards set by this list? Are any of the items too difficult or too abstract for proper understanding at the grade levels for which they are proposed? These and many other questions could be raised beneficially about the items of the Check-List and concerning other problems of grade-placement and organization of instructional materials.

We want the content of instruction to be such that it will lead to competent scholarship in mathematics. This scholarship will vary widely with different levels of learning, but the fundamental structure which gives it form is essentially the same for all levels of learning. This statement has just as significant implications for the shaping of the programs for the training of teachers of mathematics as it has for the construction of the mathematics curricula in our schools. Regardless of the level at which an individual might be engaged in teaching or the subject-matter content which constitutes his instructional medium, there are six major objectives which delineate very sharply his responsibilities in his efforts to direct the learning activities in which his pupils engage. They are: (1) Proficiency in fundamental skills; (2) Comprehension of basic concepts; (3) Appreciation of significant meanings; (4) Development of desirable attitudes; (5) Efficiency in making sound applications; and (6) Confidence in making intelligent and independent interpretations. Each pupil in school has a right to expect full opportunity for the maximum attainment of which he is capable in each of these six major objectives of learning. This expectancy on the part of the pupil defines a very specific responsibility on the part of the teacher. Each prospective teacher, in turn, has a right to expect that the training program will provide full opportunity for building up a backlog of information, appreciation, and competence in mathematics that will enable him to accept this responsibility confident that he is prepared to orient his subject properly in the perspective of the school curriculum and to deal with its properties with a feeling of security and assurance.

What is the subject matter content which gives substance to the instructional

program shaped around these six major objectives as its basic structure? The program for the training of teachers, as well as the program of instruction in our schools, finds its substance and significance in the answer to this question, for this answer will define a program for acquiring functional competence in mathematics at the various levels of instruction, and no teacher can hope to lead his students to a level of functional competence higher than that which he himself has attained. This is a basic point which, to my way of thinking, has been overlooked in previous efforts to set up a program for the preparation of teachers of mathematics. There have been several commendable efforts to construct such a program or to analyze the fundamental principles upon which such a program should be built. These studies used various methods of approach but, in the main, ended up at the same place, namely, the acceptance of the content substance of mathematics pretty much in its traditional form and then recommending a teacher-training program which consisted merely of an individualized arrangement of this traditionally organized material. This may be correct, but I wish to call attention to one very fundamental point. The evidence is rather strong that this acceptance was more nearly a tacit axiomatic acceptance than the result of thoughtful and intelligent analysis of the existent organization of mathematical subject matter in the prism of present-day demands. Why is it that we, who work in a field of thought which emphasizes the attitudes and procedures of critical thinking, should be inclined to postulate a pattern of subject-matter organization whose major cause for existence may be mere tradition? This question takes on more significance when we pause to realize that this tradition has its roots in an educational atmosphere which, at the very best, is only remotely akin to that in which our schools function today. Furthermore, its form has been fashioned largely by the counter ebb and flow of practical significance and vague abstraction. Ever since arithmetic, because of its useful techniques, was admitted to the curriculum of the early schools, these two forces have operated first to include mathematical concepts and techniques in the school program and then to modify them or exclude them entirely. There have been certain outstanding studies made of the curriculum in mathematics and the program for the training of teachers of mathematics. Among the most important of these studies are those which have been sponsored jointly or separately by the Association and the Council. There have been many changes in the mathematics program which have resulted from these studies. For example, the text which I used in 1908–09 as a high-school student of algebra did not have a single graph in it. Today graphs permeate the whole structure of algebra and arithmetic as well as other phases of mathematics. The text which I used in geometry started off with fourteen pages of definitions, postulates, and general statements, followed by Theorem I which presented the proof of the fact that all straight angles are equal. Truly an Iron Curtain of futility and "much ado about nothing" to an unsophisticated high-school junior. Today we find not only an intuitive and experimental treatment of geometry earlier in the mathematics program, but we find also that the formal study of geometry is approached through avenues which present more significant challenge and inspire a livelier interest on the part of the geometric tyro. Then there is the early use of numerical trigonometry and the recommendation for the double-track program in the early high-school years.

With the exception of the introduction of graphs and the suggestion of the double-track program, and maybe a few more such cases, all the recommendations for reform and modification of the mathematics program have been primarily within the framework of tradition. We

have seen some deletion of topics, some rearrangement of topics, the inclusion of a few new topics, and some variation in treatment. Permeating all this work there is evidence, however, of a preponderant influence of the postulates of traditional grade placement and traditional sequence. A good example of this fact is to be found in some of the considerations related to the double-track program. The literature is full of strong support for this program in mathematics, in fact there are urgent pleas for it. You will find these pleas are not for weaker standards in mathematics but for a different form of organization and presentation, one that recognizes that there are differences among people in interests and abilities. One of the major obstacles encountered in trying to give the non-sequential, non-college-preparatory track mathematical dignity is the inability to get the approval of fellow workers in the field of mathematics. This is true also at the college level. Courses that do not follow the traditional pattern of College Algebra, Trigonometry, Analytic Geometry, and the Calculus are seldom recognized for transfer value or credit. This happens frequently within the institution, or even the department, in which the course is offered. Why should this be true? What constitutes mathematical dignity? Traditional ancestry always constitutes a rather strong basis for a claim to dignity but this claim becomes very thin unless it is supported by intelligent activity and purposeful accomplishment in the present.

How do we *know* that the traditional sequence and the traditional treatment of subject matter in mathematics is the most significant possible in the perspective of problems of modern education? Maybe they are, but what authoritative evidence do we have for judging whether they are or not? When we attempt an answer to this question we must remember that with reference to any subject-matter area there are four categories into which people fall: (1) Those who are interested and have ability in the subject; (2) Those who are interested but have no ability; (3) Those who are not interested but have ability; and (4) Those who have neither interest nor ability. Furthermore, both interest and ability can range from absolute absence to very decided presence, so each of these four categories can be divided into several distinct sub-categories. In thinking about the mathematics program in our schools there is one group, a very small group, which probably should not cause us much concern; namely, those pupils who have absolutely no interest or ability in mathematics. We should feel concern for all other groups and should make sure that our thinking and planning are not prejudiced in favor of another comparatively small group; namely, those pupils who have very decided interest and ability in mathematics.

Three illustrations will suffice to give further point to the question just raised as to the significance of tradition in the arrangement and treatment of mathematical subject-matter. That these illustrations might be presented in their proper perspective, I wish to remind you that I am merely raising questions, not implying answers.

(1) At the arithmetic level of instruction attempts have been made to study grade placement of various concepts and skills. These attempts have been criticized rather severely, and there seems to be abundant reason for criticism. Nevertheless, they were efforts at an important problem and represented an approach that was willing to proceed on the assumption that maybe tradition can be in error. Why should not we who are interested in improved instruction in mathematics make some effort to study problems of this type in the same spirit of inquisitiveness? Why should not we be willing to proceed on the assumption that there might be stupidity in tradition as well as wisdom, and that it is our responsibility to winnow the chaff from the grain? It may be true that such experimentation has

brought about a "tragedy of postponement" in mathematical instruction in our schools. But how do we know, what authoritative evidence do we have for saying that one pattern of selection and organization of instructional materials is better than another? How do we *know*, for example, that the instructional emphasis on fractions should be centered in the elementary school rather than in high school, or even the college? Many of the aspects of fractions are quite abstract in nature. I have *not* been recommending that instruction in fractions be postponed until high school or college. I have been saying that there seems to be evidence that something is wrong in the present program of instruction in fractions and that we need to do something about it. It is in this context that I have raised the above questions. These same questions can be raised in connection with the grade and curriculum placement of all topics in mathematics.

(2) The literature has quite a bit to say on the pros and cons of the fusion of Plane and Solid Geometry. Careful analysis of these remarks will show that the proposals have been made and evaluated in the context of the traditional Euclidean sequence of geometrical subject matter. Maybe this is the correct organization for us to be following, but do we *know* that it is? There has been at least one interesting and significant departure from this pattern. Birkhoff and Beatley used a rather novel approach to the study of geometry in their text, *Basic Geometry*. Assuming the fundamental properties of the system of real numbers they were able to construct a two-dimensional geometry. with some incidental attention paid to intuitive three-dimensional geometry, which, to quote the authors, "although the reverse of Euclid's, is logically equivalent to it." Interestingly enough the authors make the claim that the differences between the two geometries "reflect the progress of mathematics since Euclid's time."

This work is still in the framework of the traditional separation of Plane and Solid Geometry. Maybe they should remain separated, but do we *know* whether it is possible or impossible, desirable or undesirable, from the point of view of functional learning to organize a course in geometry such that it would be an organization of materials which would present the important geometric concepts, techniques, theorems, and relationships in the more natural setting of the three-space in which we live along with their implications to the sub-spaces of one and two dimensions?

Furthermore, we might well raise the question as to whether the geometric content of the high-school mathematics curriculum is adapted to the mental and mathematical maturity of the high-school student. Euclidean geometry was formulated many centuries ago, and for mature men rather than teen-age boys and girls. Has it found its place in the high-school curriculum as a result of thoughtful educational design or accidental submergence in the fast-flowing stream of mathematical development? Questions such as these might also be asked concerning the algebraic and arithmetic content of the school program.

(3) Are we *sure* that the compartmentalized form of organization of mathematical subject matter is the most significant and efficient form of organization from the points of view of challenging instruction and effective learning? Maybe it is, but what authoritative evidence do we have for saying that it is? For example, what basis do we have for saying that it does make sense or does not make sense to say that there are at least three places in the mathematical instruction program where a general course in mathematics is desirable? Why does it not make sense to say that the beginning of the high school program (the ninth grade), the freshman year in college, and the first year in graduate school are places where the student should be given the chance to

see the significance of acquired information and skills in the perspective of the opportunities which lie ahead? A news item taken from the St. Louis *Star Times* of March 14, 1949, seems pertinent here.

> Albert Einstein, famous inventor, has poked a probing pen into education. . . . .
>
> Harking back into his teens, when he was a student at the Polytechnic Academy in Zurich, Switzerland, he writes:
>
> "In Zurich, I had excellent teachers . . . so that I really could have gotten a sound mathematical education. However, I worked most of the time in the physical laboratory, fascinated by the direct contact with experience. The balance of the time I used in the main in order to study at home.
>
> "The fact that I neglected mathematics to a certain extent had its cause not merely in my stronger interest in the natural sciences than in mathematics, but also in the following strange experience.
>
> "I saw that mathematics was split into numerous specialties, each of which could easily absorb the short lifetime granted to us. Consequently I saw myself in the position of Buridan's ass, which was unable to decide upon any specific bundle of hay. This was obviously due to the fact that my intuition was not strong enough in the field of mathematics in order to differentiate clearly the fundamentally important, that which is really basic, from the rest of the more or less dispensable erudition."

Would it not be well for the program of instruction to offer opportunities to make syntheses of previously acquired information as well as to have the opportunity to feel out the challenge of unexplored areas?

In the preceding remarks there have been no positive implications that tradition is bad and that the entire mathematical program in our schools is all wrong. Nor has there been any argument for "soft options or sagging standards." I have been making a plea for the need for setting up reasonable expectations of mathematical attainment in our democratic frame of education and within the specification of experimental evidence, and it is in that perspective that I have been trying merely to sound the caution that we should know why we are doing what we do and should be sure that we know where we are going rather than complacently following what might be merely the ruts of tradition. The spirit and intent of all remarks and questions raised have had their roots imbedded in these four pertinent questions: (1) Do we know where we are going? (2) Do we know where we want to go? (3) Do we know why we are doing what we do? (4) Do we know what we should be doing?

Is it without reason to raise the question as to the possibility that the entire mathematical program in our schools, along with the concomitant problems of the best program for the training of teachers of mathematics at each respective level of instruction, needs intelligent study by a co-operating group of educators composed of mathematicians and those representing the administrative officers and classroom teachers of our elementary schools, secondary schools, and colleges? The literature of recent years has been tragically eloquent in its appeal for such a study. The Association and Council, supported by the American Mathematical Society, should take the lead in promoting and directing this study. The study must be made in the perspective of the nature of our school population and the mathematical needs of the pupils which constitute this population. It should be a study projected on the basis of professional co-operation for the purpose of designing an educationally significant program in mathematics for our schools, a program patterned in the interest of children who are looking hopefully up the road of educational achievement rather than of specialists who are looking down this road from points safely beyond such pitfalls as indifference, conflicting interests, and learning difficulties.

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Concrete Representation of Geometric Progression
(With Illustrations from the Decimal and the Binary Number System)

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ARITHMETIC is the science of numbers. It describes what happens if certain operations are carried out, and it prescribes through symbols and signs what is to be done in certain numerical situations. Laws and rules that govern arithmetic are expressed not in words but in symbolic language, like "$a+b=c$," or, if numerical values are substituted, $5+4=9$. Such an abstract statement may be applied to concrete objects which are counted if given as individual units or measured when presented in continuous magnitudes. Five and four apples are nine apples, and five plus four feet measure nine feet. The first statement may be verified by counting the apples, the second by measuring the feet with a yardstick.

A great many of us have been led to believe that verification of abstract arithmetical statements by actual operations with concrete objects is sufficient to make numbers and numerical operations "concrete" for children. We have overlooked the fact that applications in daily life which testify to the correctness of a statement do not necessarily enlighten us about the absolute facts and truths that lie behind the symbols and signs in the world of mathematics. As Dantzig (*Number, the Language of Science*) and Hogben (*Mathematics for the Millions*) have pointed out so clearly, numbers and signs constitute the arithmetical language and describe relationships and operations in a kind of shorthand. We must be able to visualize what the symbols describe; we must look behind them and through them to understand fully the *structure* of the arithmetical relationships with which we are concerned. To a true mathematician, the operations with figures are clearly visible and his figuring with symbols is concrete; he actually sees with his mental eyes how a series increases, what an equation indicates, and so on.

Structural Arithmetic is a new approach to the teaching of arithmetic with the goal of representing numbers by devices which have all the properties of abstract numbers. These devices allow pupils of all ages to carry out with their own hands and see with their own eyes what figures and signs tell them. Just as a teacher of French points to a chair when naming it *la chaise*, so the teacher of Structural Arithmetic may point to blocks or cubes or other devices and say: What you are seeing here is expressed by $5+4=9$, or, by $1+10+10^2+10^3$, or, by $1+2+2^2+2^3+2^4+2^5$, or, by carrying 1, etc.

In this article, I will attempt to show how the structure of the decimal system, the binary system, geometric progression, and the structural role of "carrying 1" can be shown with the proper devices of Structural Arithmetic. Let us see how a pupil may get first-hand experience with our number system, based on the powers of 10, and how he discovers the dramatic happenings which lie behind "carrying 1" if you will only unveil its secrets for him and go behind the strictly mechanical rules.

The abacus, which once again is enjoying a certain degree of popularity in schoolrooms, is sometimes used in an attempt to make the number system concrete. Demonstrations with an abacus show children that if certain mechanical rules of operation are followed, the correct answer comes out in the end, just as a piece of candy comes out of an automatic vending machine when a coin is put in the proper slot according to the directions. However, the general acceptance of the Hindu-Arabic system of notation four centuries ago has made it

possible to use figures in rows on paper in a more direct way than was ever possible with the abacus. For example, add 1 to 999:

$$\begin{array}{r} 999 \\ +\quad 1 \\ \hline \end{array}$$

Here we see 9 in each row (we do not have to count beads as in the abacus), and if we apply the rule that ten 1's have to be exchanged by 1 ten, etc., it is much simpler to work with symbols than with beads. Unfortunately, children today are usually taught to use figures in adding just as mechanically as calculators move beads on the counting frame of the abacus. Listen to what a child says in solving the above example: "Nine and one are ten; put down zero, carry one. Nine and one are ten; put down zero, carry one. Nine and one are ten; put down zero, carry one." He writes the figures in the answer:

$$\begin{array}{r} 999 \\ +\quad 1 \\ \hline 1000 \end{array}$$

He reads the result as "one thousand" just as the calculator records the total to be one thousand when he sees just one bead in the fourth row of the abacus.

What does the pupil *see?* Nothing but figures (or beads). What does he *think?* Only that he is not to forget to "carry 1." Instead of a fascinating number experience which provides pleasure and even thrill, the pupil stares blindly at his example (or at his beads in the counting frame), and he neither sees nor enjoys the experience that the operation has given him.

Let us study the same example with the materials of Structural Arithmetic. Our decimal system is built on the powers of ten. That means we have to deal with 10^0 or one, 10^1 or ten, 10^2 or one hundred, 10^3 or one thousand, and so on. In Structural Arithmetic, the pupil is given one unit, or cube, to stand for 10^0. Ten of these cubes equal a Ten-Block, which is ruled into 10 cubic units and thus seems constructed of ten cubes apparently "glued" together. This Ten-Block represents 10^1, or ten. There are 10 such Ten-Blocks that equal the next higher denomination, the Hundred-Square, which represents 10^2. It shows 10 tens fastened together, or 100 cubes. And 10 such Hundred-Squares put one on top of the other equal the Thousand-Cube that represents 10^3.

Any number between 1 and 1,999 may be constructed with these materials. The number 1,001 would show a Thousand-Cube and a Unit Cube. The number 1,350 would be constructed of 1 Thousand-Cube, 3 Hundred-Squares, 5 Ten-Blocks and no Unit Cubes. Let us construct the number 1,111. (See Figure 1.)

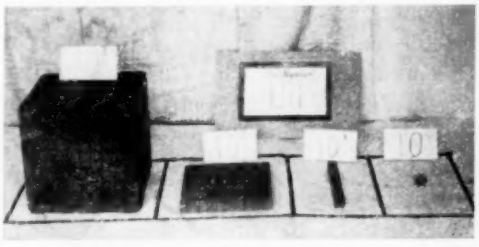

FIG. 1. The Number 1,111.

This is the number one thousand one hundred and eleven as built from 1 Thousand-Cube, 1 Hundred-Square, 1 Ten-Block and 1 Unit Cube. It is simplicity itself to show the pupil how each power of ten is expressed by a figure in a given *position*. One look at the materials shows the different value of each 1 in the number 1,111. It means a huge Thousand-Cube, a Hundred-Square that equals the bottom layer of the cube (being 1/10 of 1,000), a 10-row (or Ten-Block that is but 1/10 of the Hundred-Square), and, finally, a Unit Cube (that is 1/10 of the 10-row). And, inversely, we see that each figure is ten times as great as the figure to its right. The picture thus shows the first four terms of the geometric progression according to which the decimal number system is built:

$$1+10+10^2+10^3.$$

Now, let us construct the number 999.

(See Figure 2.) It is built up from 9 Hundred-Squares, 9 Ten-Blocks and 9 Unit Cubes.

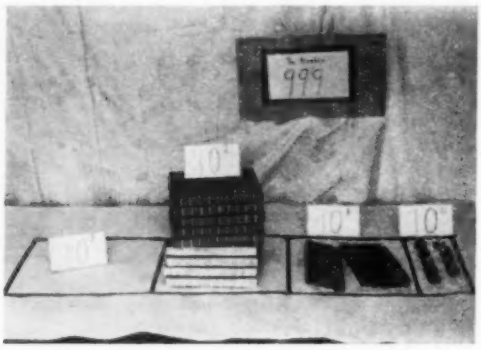

FIG. 2. The Number 999.

Add 1 cube, and the most dramatic happening ensues which only a movie can adequately show in operation. That 1 cube added to the 9 Unit Cubes makes 10 cubes. We may exchange 10 cubes for the equivalent Ten-Block which we actually carry to the next place. The 9 tens which are already there and this added Ten-Block form a Hundred-Square which is now substituted for the 10 Ten-Blocks. This Hundred-Square has to be carried to the next position, where it belongs. If we put this Hundred-Square on top of the pile of 9 Hundred-Squares, the figure of the Thousand-Cube emerges. Therefore, we carry again, this time the Thousand-Cube which we have substituted for the 10 Hundred-Squares. Thus, all the places up to the Thousand-Cube are emptied. (See Figure 3.)

When our pupils engage in this fascinating number experience, they often

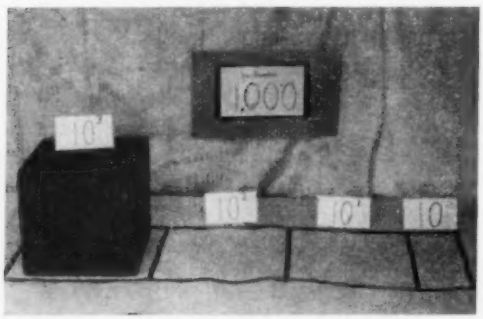

FIG. 3. The Number 1,000.

exclaim excitedly, "Carry! Carry! Carry! It gives the Thousand-Cube! It makes just 1,000!" The recording of this concrete operation is "999+1=1,000." If, later, the column form of recording is used, the carrying on paper is full of meaning. The children actually handle tens or hundreds or thousands just like units. However, they are fully conscious of how different the pieces are; form and size vary if a number is moved up the decimal scale, as 9, 90, 900. The numbers are not represented merely by 9 beads of the same size in different positions, but by 9 Unit Cubes, or 9 Ten-Blocks, or 9 Hundred-Squares. Do children understand the significance of these cubes and blocks of different sizes? Of course! Just try to give a child 1 candy in a different "position"—or wrapping—instead of 10 candies of the same size. Why should he exchange the 10 candies for only 1 just because of the different wrapping? Then, ask him whether he would rather take home 1 chocolate bar consisting of 10 squares or 10 separate squares of the same size. He sees clearly that each offer consists of the same amount of candy, but he will probably take the bar for convenient transportation. The exchange of 10 beads in Units place for 1 bead in Tens place is an adult convention which a child may learn to carry out obediently, but which does not make sense for him, since 1 bead and 10 beads of the same size will never look interchangeable.

With the materials of Structural Arithmetic, "borrowing" may be shown just as concretely as "carrying." The same vivid impression is gained if the Thousand-Cube is presented and the pupil is asked to take 1 from 1,000. A child often puts his finger on one unit of the Thousand-Cube, and says, "We can't break a cube off!" So, we exchange the Thousand-Cube for 10 Hundred-Squares, but, still we can't break one cube off. One Hundred-Square is replaced by 10 Ten-Blocks, but, again, no single cube can be taken away. Now, the Ten-Block is exchanged for 10

single cubes. We can easily take 1 cube off! The result, of course, is 9 Hundred-Squares, 9 Ten-Blocks, and 9 Cubes, or the number 999. The whole operation is recorded as "1,000 − 1 = 999."

In the Hindu-Arabic system of numbers, we have only the symbols from 1 to 9 and the zero as place holder to represent any number we may think of—billions and trillions, or millionths and trillionths. In Structural Arithmetic, we indicate the continuity of the number system in going beyond one thousand to the larger numbers and downward from 1 to show decimal fractions, thus extending the series in both directions. If we take 1 cube and slice it into tiny squares, we get 1/10 of the Unit Cube, or the tenths. If we cut these into miniature Ten-Blocks, we get 1/100 of the Unit Cube, or the hundredths. And if we cut one of these small Ten-Blocks into 10 miniature cubes, we arrive at 1/1,000 of the Unit Cube, or the thousandths. These blocks are easy to construct. Thus, we may build with the materials of Structural Arithmetic any number between one thousand and one thousandth, or, any number between 10^3 and 10^{-3}.

Let us look at the geometric progression which represents the decimal number system:

$$1+10+10^2+10^3+10^4+10^5+10^6+\cdots.$$

Of this progression, we have shown the first four terms with 1 cube, 1 Ten-Block, 1 Hundred-Square and 1 Thousand-Cube. (See Fig. 1.) The sum of $1+10+10^2+10^3$ is the number 1,111 which we have shown above. Must we stop here or could we possibly go beyond 10^3? At first sight, it seems impossible. How could we ever imagine a concrete representation of 10^4, 10^5, 10^6—how could we render visible the fourth, fifth, or sixth dimension!

In order to solve this problem, we had but to visualize how we go from one denomination of 10 to the next higher one by simply multiplying by the base 10. If we multiply 1 by 10, it means that we allow the 1 to grow in length until the full 10-row is reached. If we multiply 10 by 10, it means that we let the 10-row grow in the second dimension until we arrive at the 100-Square with the area 10×10. If we multiply once more by 10, the 100-Square grows in the third dimension, i.e. into the Thousand-Cube with the volume of $10\times10\times10$. We had but to go on in this way: the next step is to handle our Thousand-Cube as a "giant unit cube" and take it ten times; the result is a giant ten-row of Thousand-Cubes representing ten-thousand. Ten of these form a giant Hundred-Square consisting of 100 Thousand-Cubes representing one hundred-thousand; and, finally, ten layers of these squares form another cube which the children called the "Gigantic Cube" because it contains a thousand Thousand-Cubes. Its name is one million.

This concrete representation of 10^4 or 10,000, of 10^5 or 100,000, and of 10^6 or 1,000,000 is in accordance with the nomenclature (ten-thousand for 10^4, one hundred-thousand for 10^5) as well as with the grouping of numbers within periods. We talk about the "Unit Period" which consists of 1's, 10's, and 100's of units. We switch to the "Thousand Period" which contains 1's, 10's, and 100's of Thousands. We may go on to the "Million Period" in which we find 1's, 10's, and 100's of millions. We remain in the first, second, and third dimension, switching simply to new units of measure. We have the sequence of cube, row, square—cube, row, square . . . ", and, consequently, we are able to construct a visible image of 10^4, 10^5, 10^6, and so on.

Instead of a concrete representation of 10,000 or 100,000 or 1,000,000 in the decimal system, we shall show these further terms of the geometric progression concretely by working with a number system with a base smaller than ten. Any number system which is constructed according to the powers of its base (b) shows the structure of a geometric progression:

$$1+b+b^2+b^3+b^4+b^5+b^6+\cdots.$$

In place of ten, the *binary* system is built on the powers of two. Thus, b^6 would mean 2^6. Whereas b^6 in the decimal system must have the size of 1 million of our unit cubes, we need but 64 unit cubes to construct the number 1,000,000 in the binary system. Before showing the photograph of this small "gigantic cube," we have to study the notation in the binary number system.

The binary number system has a base of 2, and any number we may think of is recorded with just the two symbols 1 and 0. A 1 in units place means one unit, a 1 in the second place means 2, a 1 in the next place 2^2, and so on. If we look at the figures in the decimal system, each position characterizes a value ten times as much as the preceding one; in the binary system, it is twice as much. It is hard to talk about 100 in the binary system and to visualize that it means 2^2, or 4, if we still give "100" the name "hundred", which stands for ten tens and not for two twos. For clarity, we shall give the figures 10, 100, 1,000, etc., names which show that we are working with the binary system, or with 2, 2^2, 2^3, etc. Let us call "10" in the binary system "twin," instead of ten; 10^2, or "100," will be named "twindred," 10^3, or "1,000," will be one "twosand"; logically, then 10^4, or "10,000," must be called "twin twosand"; 10^5, or "100,000," "twindred twosand"; 10^6, or "1,000,000," one "twillion." We would go on with "twin twillion," "twindred twillion," etc.

In our decimal system, we find as many basic addition facts as there are possibilities to add the figures from 1 to 9 to one another and to each number itself. In the binary system, we have but to learn that "1+1=10," or, "one and one equal twin." The rest is done by "carrying." In both systems, one has to understand the "zero-facts," i.e., that any number to which zero is added equals itself. Here are a few addition examples in the binary system; first let us examine the series of "adding 1" which shows us how we count in the binary system:

Binary	Decimal
0+1= 1	1
1+1= 10	2
10+1= 11	3
11+1= 100	4
100+1= 101	5
101+1= 110	6
110+1= 111	7
111+1= 1,000	8
1,000+1= 1,001	9
1,001+1= 1,010	10
1,010+1= 1,011	11
1,011+1= 1,100	12
1,100+1= 1,101	13
1,101+1= 1,110	14
1,110+1= 1,111	15
1,111+1=10,000	16

Thus, the number series from 1 to 16 in the decimal system corresponds to the numbers from 1 to 10,000 in the binary system; counting 1, 2, 3, 4, 5, · · · , 16 corresponds to: one, twin, twin-one, twindred, twindred and one, · · · twin twosand. "Adding 2" in the binary system is adding 10 to any of the numbers; "adding 3" means adding 11, and so on.

The multiplication table in the binary system contains the basic fact "1×1=1"; positional notation takes care of the work if we have to multiply 11,110 by 111, and we must not forget how to carry if we add the partial products. If we multiply by 10, we multiply by the base. In the usual terms, we "annex zeros"; in our terms we change a number to the next higher denomination.

$$10\times 1 = 10 = 10^1$$
$$10\times 10 = 100 = 10^2$$
$$10\times 100 = 1000 = 10^3$$
$$10\times 1000 = 10000 = 10^4$$

Translated into the decimal system, "times 10" in the binary means *doubling*, since we are dealing with a base of 2. 10, or twin, is 2; 100, or twindred, is 4; 1,000 or twosand, is 8; 10,000, or twin twosand, is 16; 100,000 or twindred twosand, is 32; 1,000,000, or one twillion, is 64; and so on. The geometric progression which shows the structure of the binary system is

$10^0+10^1+10^2+10^3+10^4+10^5+\cdots$ or,

$$2^0+2^1+2^2+2^3+2^4+2^5+\cdots$$

in the decimal system.*

Let us construct this progression with the materials of Structural Arithmetic. We have the sequence of cube, row, square, cube, row, square, and so on. Look at the next photograph which shows this progression and represents the number 111,111 or twindred twin-one-twosand, twindred twin-one. We built it simply by putting 1 cube in first place; 2 cubes (in a row) in second place; 4 cubes in a square next; then a cube of 8; a 2-row of 8, or 16; a 2-square of 8, or 32. (See Fig. 4.)

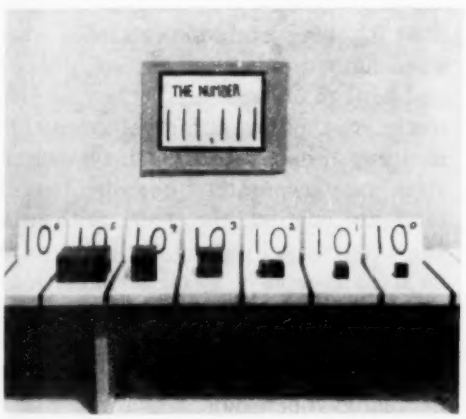

FIG. 4. The Number 111,111 in the Binary System.

Once more, we see how the "periods" repeat themselves, always using the cube as a new unit of measure. We have the "Ones Period," called here, ones, twins, twindreds. The "Twosand Period" follows with one twosand, twin twosand, twindred twosand. Then comes the "Twillion Period" with one twillion, twin twillion, twindred twillion, and so on.

We have seen how impressive an operation resulted when we "added 1" to the number 999 in the decimal system. As a parallel operation, let us "add 1" to the number 111,111 in the binary system.

* In order to show the parallel structure between the two systems, the indices are given in the familiar decimal notation even in the binary example.

How dramatic the scene is in action, yet so uninteresting when figured on paper without understanding.

$$\begin{array}{r} 111,111 \\ +\quad\quad 1 \\ \hline 1,000,000 \end{array}$$ Remember: $1+1=10$

Done mechanically, as our children are usually taught arithmetic, the figuring would follow this scheme: One and one are twin; write the zero, and carry 1. 1 plus 1 are 10; write 0, and carry 1, etc. The result is the number 1 twillion.

In Structural Arithmetic, we add 1 cube to the 1 cube already at that place. It makes 2, or twin, and has to be carried. That gives two twins—to be carried again to the 100's place. A Twindred-Square added to the Twindred already there gives two of the Twindred-Squares—which have to be carried as a second cube to the 1,000 place. We can't have *two* Twosand Cubes in the 1,000's place, so we have to carry again, and so on. Carry, carry, carry, until we have emptied all the places and have left only a 1,000 cube of 1,000 cubes, or 1,000,000, or 1 twillion! (See. Fig. 5.)

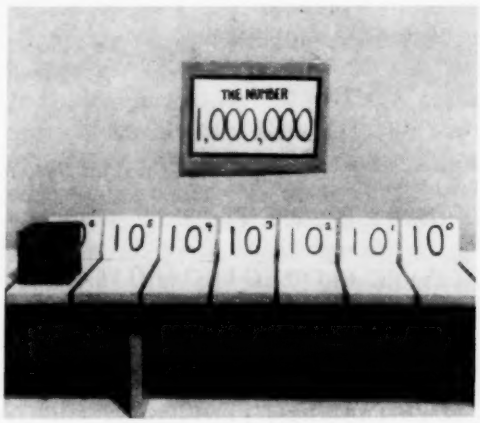

FIG. 5. The Number 1,000,000 in the Binary System.

These two parallel cases of "adding 1" to the number 999 in the decimal system or to the number 111,111 in the binary system, are instances of the same formula In any geometric progression with a base

(b), and whose first term is 1, we find the sum (S) of the first n terms as:

$$(1) \qquad S_n = \frac{b^n - 1}{b - 1}.$$

Let us multiply both sides of the equation by $(b-1)$:

$$(2) \qquad (b-1)S_n = b^n - 1.$$

Now add 1 to both sides of the equation:

$$(3) \qquad (b-1)S_n + 1 = b^n.$$

This last formula (3) states the following rule for all possible number systems of any base that are built as geometric progressions: If you sum up n members of the progression and multiply this sum (S_n) by $(b-1)$, the adding of "1 more" results in a total of b^n.

In the decimal system, the base is ten and the "base less one" $(b-1)$ equals 9. If we sum up the first three terms of the progression in the decimal system, we get

$$S_3 = 1 + 10 + 10^2$$
$$= 1 + 10 + 100$$
$$S_3 = 111.$$

Formula (3) for this case states:

$$9S_3 + 1 = 10^3$$

or,

$$(9 \times 111) + 1 = 1{,}000$$
$$999 + 1 = 1{,}000 \ (\textit{Decimal System})$$

(See Fig. 2 and Fig. 3.)

If we substitute $b=2$ in the same formula, the "base less one" $(b-1)$ is 1. If we sum up the first six terms of the progression for the binary system, we arrive at

$$S_6 = 1 + 10 + 100 + 1{,}000 + 10{,}000 + 100{,}000$$
$$S_6 = 111{,}111$$

Formula (3) states for $(b-1) = 1$

$$S_6 + 1 = 10^6$$
$$111{,}111 + 1 = 1{,}000{,}000 \qquad (\textit{Binary System})$$

(See Fig. 4 and Fig. 5.)

We have seen in the pictures an indication of the operation in which "adding 1" to 999 in the decimal system or to 111,111 in the binary system results in "carry, carry, carry . . . " until all the originally occupied places are swept clean. The total is always a number in the denomination which is just 1 *beyond* that of the last term of the progression: a thousand in the first example and a twillion in the second example.

We could work just as easily with a *quinary* number system and add 1 to a number like 4,444. The total must be 5^4, or 10,000 in the 5-system.

The "visible" representation of any number system makes arithmetical operations thrilling and unforgettable which, when figured out on paper, are often dull and hard to understand. Structural Arithmetic gives pupils an actual grasp of the meaning of operations which symbols and signs may accurately describe but also tend to conceal the true arithmetical characteristics. First-hand experiences unlock the secrets of mechanical rules, and such work as "carrying" on paper is done quickly and efficiently. For our children, it is the recording of a fascinating, understandable operation which they have carried out with their own hands.

Man has made his ten fingers the measure of "manyness" or plurality. He did not know then that such simple grouping of objects into tens and "tens of tens" was the first ingenius step to the invention of a number system based on the powers of ten. An unknown Hindu genius invented "positional notation" according to which the place of a number indicates its value, telling us whether we are dealing with 1's, 10's, 100's, 1,000's, and so on. This made a pure arithmetical step into a geometrical relationship: a number system built on the powers of its base forms a geometric progression. Man did not fully grasp what tremendous achievement is represented in his number system; and, unfortunately, it's unique beauty is not brought to life in today's classrooms.

Square Root

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The subject of square root cuts across the entire range of mathematics courses, from arithmetic to the calculus, and seems to cause trouble in every course.

Traditionally it is taught by a method stemming from the binomial theorem which involves remembering several rules, all quite mysterious to most students. While much has been written about simplifying this method and improved methods of teaching it, little or nothing has been done toward replacing it by a method fully as time-honored and much more intelligent.

In the usual algorithm, for each successive digit in the root, it is necessary to multiply by 20 the digits of the root found up to that point. This rule can be justified to an algebra class, and explained to a bright arithmetic student, but neither will remember it. Every teacher has had the experience of being asked to explain square root to someone who learned it at one time and has since forgotten it.

Ask the student who wants to learn, or relearn, square root what it is that he is looking for. Eventually he will furnish the definition: a number which can be multiplied by itself to produce the given number. This definition furnishes a better method.

Consider an example: the square root of 103,041 is desired. If we knew the answer exactly, then simple division would show divisor and quotient identical:

$$\begin{array}{r|l} & 321 \\ \hline 321 & 103041 \end{array}$$

Therefore, to find the root, suppose we guess at it, using any information we have, and divide. We are free to borrow a bit from the traditional method; that is, starting from the decimal point we may separate this number into periods of two digits to ascertain that the root must have three digits (in this example) before the decimal point, and that it is clearly greater than 300, whose square is 90,000. Suppose that our first guess is 310. Simple division, to as many digits as our guess, gives:

$$\begin{array}{r|l} & 332 \\ \hline 310 & 103041 \end{array}$$

and it should be clear that 310 is too small, since it "goes into" the dividend 332 times; similarly 332 is too large. The correct root must, then, lie between these two numbers and, lacking any better information, the second approximation should be half way between the two. Interpolation between 310 and 332 (half of their difference added to the smaller number) gives 321, which would be the second "guess" toward the root. Another such division will establish its accuracy.

This method was developed by Hero of Alexandria over 2000 years ago and is known as the Heroian algorithm. Hero showed, in fact, that if pq equals N, where p is not equal to q (if p equals q, then either is the desired root, of course), then any number d such that $p<d<q$ will lead directly to the square root of N, and in particular $d=\frac{1}{2}(p+q)$ is the simplest number with which to work. Hero worked entirely in fractions; our decimal notation is not only faster and more convenient, but the principle involved is much clearer.

How much must be remembered with this method? Just two things: the definition of what it is we seek, and the concept of interpolation. Obviously, the better the initial guess, the quicker the solution.

As another example; let us extract the square root of 10 by this method. As a first guess we can use 3; the first quotient

is then $3\frac{1}{3}$, and interpolation between 3 and $3\frac{1}{3}$ gives $3\frac{1}{6}$ (the first steps here can be done well in fractions, mentally). 10 divided by $3\frac{1}{6}$ equals 60/19, or 3.158. The root now lies between 3.167 and 3.158. The interpolation is

$$\begin{array}{r} 3.167 \\ 3.158 \\ \hline 2 \;\big|\; .009 \\ \hline .0045 \end{array}$$

and 3.158 plus .0045 equals 3.1625, which is correct to four significant digits.

The algorithm is especially adapted to machine computation (it is, in fact, the method which the makers of desk calculators publish for the extraction of any square root.) Consider its ease and speed for the computation of the square root of 5, using a desk machine. The first steps may be mental. We try 2 as a first approximation; 5/2 equals 2.5, and interpolation between 2 and 2.5 gives 2.25. Now, using the machine, 5/2.25 equals 2.222 and interpolation gives us 2.236, which is correct to four significant digits in one machine operation. If, now, we divide again, using 2.236 as our divisor, the quotient is 2.23613595, and interpolation gives 2.2360680. This result is correct to *eight* significant digits. One more such division with a standard desk calculator would yield, after interpolation, a result correct beyond the capacity of the machine. The entire operation described takes, with practice, only a few seconds.

Provided the initial guess is not too wild, this algorithm is fully as fast as the method usually taught in our schools. Indeed, we can show that the relative error, after interpolation, is generally less than the square of the relative error in the larger of p and q. This effectively doubles the number of significant digits which repeat in divisor and quotient. For example, if 1.414 be used as a first approximation to the square root of 2, the first quotient will be 1.41442715. Since 4 digits repeat in divisor and quotient, it is proper to interpolate 4 digits more, giving 1.4142135, which is correct to eight significant digits.

The method is accurate, being an iterative process; in fact, it automatically corrects errors, since an error in division means only that the next divisor is not as accurate as it might be. It is intelligible; there is nothing mysterious to be explained or justified. It is intelligent; it can be recalled after long periods of time. It is of particular value when it is desired to extend a known root to more decimal places. For example, if the square root of 5 is needed to more places than are given in a table (2.236), then this tabular value is the first approximation and the computation immediately extends the root without these first digits being re-computed; in general, the number of significant digits will be doubled.

Note that the algorithm offers no difficulty to the student in the extraction of roots of pure decimals. The fact that the root of a fraction is greater than the fraction itself usually causes difficulty. If the first approximation, using Hero's method, should overlook this fact, the method will quickly reveal the error.

It should be pointed out to students that there are many methods of extracting square root. Besides slide rule and logarithms, Horner's method of successive approximation can be applied to the equation $x^2 = N$. Or, it is possible to interpolate directly in a table of roots. For example, using a table of roots of numbers from 1 to 1000, the square root of 234.6 can be found by interpolating between the values given for 234 and 235 and adjusting the decimal point. For numbers which are close to unity, the square root may be found by halving the decimal part of the number; e.g., the square root of 1.00064 is 1.00032 and the square root of .999964 is .99982. The binomial series will give us square roots (particularly those of numbers lying near to a perfect square, such as 26). The method of subtraction of odd numbers is

also widely used on desk calculators.

The method outlined in this paper is usually credited to Newton, since it is, in fact, a special case of Newton's approximation method for the solution of equations. The method readily extends to higher roots, the n^{th} approximation to any root being given by the formula:

$$A_n = \frac{1}{i}\left[N/A_{n-1}^{i-1} + A_{n-1}(i-1)\right]$$

where i is the desired root and N is the number whose root is sought.

The advent of high speed automatic computing machinery makes this method of especial interest, since it is the method which these machines use. The first guess is usually taken as 3 (or 30, or 300, etc., depending on the position of the decimal point) arbitrarily, and the machine iterates until two successive trials agree to the number of digits desired.

The "Linear" Equation

By A. R. JERBERT
University of Washington, Seattle, Washington

IN THE equation,

(1) $$3x+y=6,$$

if

$$x=-1,\ 1,\ 4,$$
$$y=9,\ 3,\ -6.$$

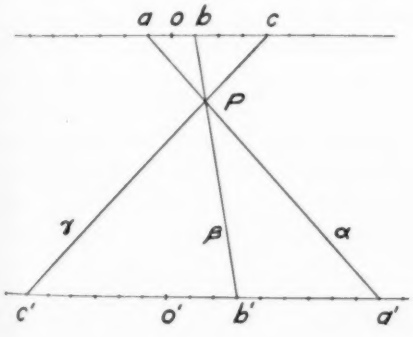

FIG. 1.

In Fig. 1 the line segment from the origin to a, which is identified with a, represents the value $x=-1$. a' marks the corresponding value $y=9$. In like manner b, b' and c, c' picture, respectively, the paired values $x=1, y=3$ and $x=4, y=-6$.

If the origins of the two scales are superimposed and the y-axis is rotated 90° we obtain Fig. 2 below.

In Fig. 2, $a(x=-1)$ and $a'(y=9)$ determine a point A* as the intersection of a horizontal through a' and a vertical through a. Thus,

$$A = a,\ a',$$

in the sense that the simultaneous values $x=-1$, $y=9$, pictured by a and a' are

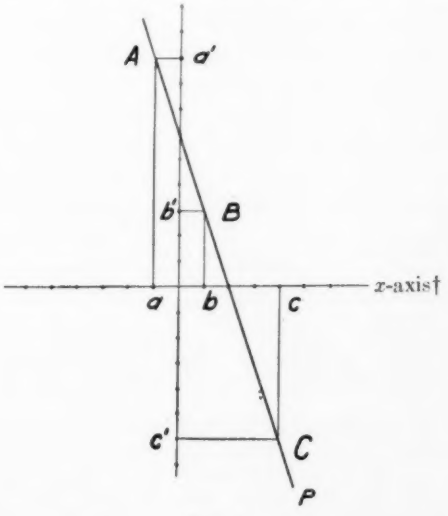

FIG. 2.

* $x=-1$, is called the abscissa (ab = away, scissa = cut) of A because it is the segment "cut off" the x-axis by the vertical line through A. Thinking of $y=9$, as the segment aA we refer to it as the ordinate (*order*, rank, elevation) of A.

† In the perpendicular arrangement the number scales are referred to as axes. $+y$ values which are to the right of O' in Fig. 1 are *above* the common origin in Fig. 2.

concentrated in the single point A. B and C serve similarly in place of the pairs b, b' and c, c'. In a dual manner we can think of the point pairs a, a', b, b', c, c', in Fig. 1, as equivalent to the respective connecting lines α, β, γ.

In Fig. 2 the points A, B, C, lie upon a line p and in Fig. 1 the lines α, β, γ, intersect in a point P. Moreover any x and y values which satisfy equation (1) simultaneously are coordinates of a point which is on the line p. Conversely, any point on p has such coordinates.‡ The line p is thus the geometric image or graph of equation (1). Since similar statements hold for the point P we can with equal right refer to it as the "graph" of equation (1).

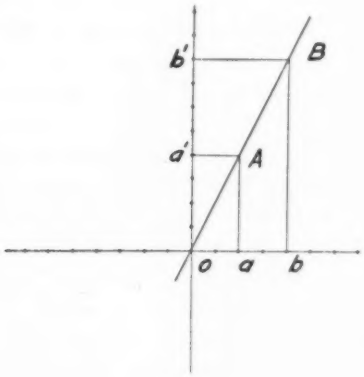

Fig. 3.

That an equation of the form,

$$ax+by=c,$$

has a straight line (or point) graph is a surprising and possibly intriguing§ discovery but at the expense of a clear understanding. Suppose, however, that we begin with a very simple equation such as,

$$y=2x.$$

In Fig. 3 the 2 to 1 ratio of the ordinates aA, bB, to the abscissas Oa, Ob, makes it obvious that the points A, B, C, lie on a straight line through the origin whose slope is 2.

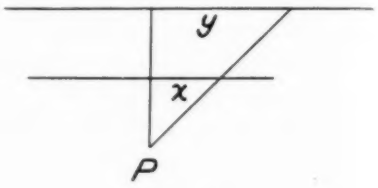

Fig. 4.

In a dual manner lines drawn through a point P (Fig. 4) whose distances from the y and x scales are in a 2 to 1 ratio, evidently cut off corresponding y and x segments which are in the same ratio. In either Fig. 3 or Fig. 4 the transition to $y=mx$, offers little difficulty after working with numerical instances of the slope constant m.

In the general case,

$$y=mx+b, \tag{3}$$

it is easy to see in Fig. 3 that the effect of the b is simply to move the graph b units measured along the y-axis. Writing equation (3) in the form,

$$y=m(x+b/m), \tag{4}$$

we see that in Fig. 4 the effect of the b term is to shift the x origin a distance b/m.

If the y term is present ($b\neq 0$) equation (2) can always be solved for y as in equation (3). If the x term is also present ($a\neq 0$, $b\neq 0$) we can solve for y as in equation (4).

‡ The formal proof of this comes much later, if at all.

§ Recalls the response of a Junior High miss, who was asked how she liked algebra. "Fine, they do such funny things with funny numbers."

Contributions of the Activity Movement to the Teaching of Arithmetic

By Lois Knowles
University of Missouri, Columbia, Mo.

Has the activity movement which had its beginning some fifty years ago and which has been discussed, tried out, and evaluated under many different conditions made any impression on the practices of teaching arithmetic in our public schools? If progress has been made in techniques, materials, and the selection of content in arithmetic, is it due in any way to the philosophy underlying the activity movement? Finally, what are some of these changes?

A brief review of the program for the half-century may help in understanding the activity movement and evaluating its contributions. Dr. Dewey's work, begun at the University of Chicago in 1896, emphasized the "reconstruction or reorganization of meaningful experience by means of creative intelligence applied in purposeful activity." The Laboratory School at the University of Missouri in which I work was organized by Dr. Meriam in 1904 on this philosophy. The four types of activities which were developed in this school were observation, play, stories, and handwork. I believe that arithmetic was purely incidental in the early program.

Much experimental work has been done since 1900, and we owe a great deal to the progressive men and women such as Dr. Dewey, Dr. Meriam, Dr. Kilpatrick, Dr. Jersild and others who have helped us to focus attention on children and the ways in which they may be expected to grow, behave, and learn in every experience, including arithmetic.

A study of practice in the activity schools of the past leads one to believe that perhaps arithmetic was subordinated to other fields of learning. The phrase "incidental theory of arithmetic teaching" crept into the terminology of the activity movement, and received a great deal of condemnation through the years. During the early period I believe that arithmetic was not talked about as much as the content of some other areas, particularly the language arts and the fine arts, in which the emphasis was placed not on conventional subject matter, but on freedom, spontaneity, and creative expression. Without a doubt these factors had an effect on arithmetic, but perhaps in rather an indirect way.

The so-called activity school has received much adverse criticism, some of it quite justly. On the other hand the emphasis which the activity movement has given to the ways in which the child reacts to his environment, the way he behaves when confronted with new situations, and the teacher's part in helping him to decide on a course of action for which he is willing to be responsible—these emphases have contributed a great deal to the practices which are used in the elementary school of today. The teacher with this philosophy uses discussion, visual aids, excursions, etc., and selects from the variety of teaching materials those which help the child to make generalizations, to set up a method of thinking, to define for himself the problems which confront him and to find satisfactory ways of solving these problems. In a school organized on this philosophy the pupil sees meaning in his activity, he has a sense of its value for him, and he has to this extent enlarged and gained control over his immediate world.

This description of a modified activity school—the child-centered school of today—makes one distinctly conscious of the fact that it is quite different from the first purely activity school. From a beginning in which the center was on child-interest

alone, there has been a re-focusing to include the results of research to improve learning: an integration of the child's interests, the best materials available, and the very best techniques of presentation and evaluation.

The activity movement and the development of the pupil-center-of-interest type of school, has been paralleled and identified with the study of children, their interests, their growth patterns, the ability of children at different stages of growth, variations of ability in different individuals, etc. This effort to approach the goals of education from the inside, rather than from the outside, of the child has been one of the major accomplishments in elementary education. Child-study, the findings of psychologists, and the experimental programs for improving instruction have resulted in the principles and practices which most elementary teachers know and respect as readiness, maturation, individual differences, development of personality, integration, democratic techniques, group consciousness and the like. For the past ten years the literature in elementary education has been full of statements to the effect that we are teaching children not subject matter.

If the schools of the present and of the future are to make the best use of these results, the future of arithmetic should be commensurate with the future of any learning activity at the elementary level. I shall mention some of the aspects of arithmetic teaching which have been clarified by practices in recent years.

We may think of arithmetic as one of the areas in which we help children gain control over their environment. When a child has learned to use the clock to tell time, he can adjust his activities so that he can make the bus, join his friends at a certain time, get to school, and otherwise regulate his immediate life. This quantitative area extends and enlarges as the child grows and develops, but his reason for learning, his way of learning it, and his attitude toward learning is basically the same in arithmetic as in anything else. The content of arithmetic is not thought of so much as subject-matter-to-be-learned as a field of learning in which the child is helped to enlarge his world. Mrs. Mossman, in her book *The Activity Concept* says, "If we ask what makes subject matter essential or fundamental we shall eventually agree that it is value in living for the one who learns it." A felt need on the part of the child results in a desire and satisfaction in learning.

It is also believed that children can assume a great deal of responsibility for making intelligent use of their abilities. This is just as true in arithmetic as in anything else. Those who have used the self-instruction methods which Dr. Wheat and others have advocated believe that the independence and self-direction which result are examples of much-to-be-desired democratic techniques.

The literature of arithmetic is full of suggestions on how to make arithmetic meaningful. The use of concrete materials and visual aids and the deliberate emphasis on directed functional learning are examples of these techniques. Giving a child the techniques of how to find out what he wants to know is just as important in arithmetic as in any other field. The child who learns by manipulative discovery and by forming generalizations, has at hand a technique of finding out what he wants to know. This is in contrast to the pupil who learned by rote memory and relied on memorization.

The activity movement has also given emphasis to the interrelationship of ideas in different areas of learning. The understanding of new ideas often requires the interpretation of a quantitative description. Relational thinking is demanded. A child looks up from his reading and asks, "It says here, 'a party of'—how many is a 'party'?" A quantitative interpretation is demanded. Ideas such as these are presented many times a day in any group of children.

An examination of arithmetic textbooks, manuals, and teaching aids shows that changes have taken place in commercial materials. Teachers are given much help in developing a meaningful approach to arithmetic. They are supplied with materials and with suggestions for making arithmetic functional.

In conclusion, it should be pointed out that the activity school has contributed a great deal to our present philosophy and practice in the elementary school. The traditional school has taken on many of the attributes of the activity movement, and the activity school has been modified by research and experimentation. This change, or evolution, in both types of schools is good to see. For as a result of these changes our children are deriving greater benefits from their school experiences.

Teaching Plane Geometry to Mentally, Physically, and Emotionally Handicapped Pupils*

By Allene Archer

Thomas Jefferson High School, Richmond, Virginia

Thomas Jefferson High School in Richmond, Virginia, is located in a "better class" neighborhood. For the most part its pupils come from homes above average in culture, education, and income; and between 50% and 60% of its graduates enter college.

For some time the teachers of plane geometry have been concerned about the number of failures in their classes and have tried several plans in an effort to improve the percentage of promotions. The failures fall into three groups—the mentally, physically, and emotionally handicapped pupils. About 84% of these pupils have I.Q.'s below 110, about 48% below 100, and about 13% below 90. Some have been victims of polio, rheumatic fever, scarlet fever, and other diseases which leave an after effect. Some of them are crippled and find writing difficult and many lack muscular coordination. Many are from homes disrupted by divorce, death, or severe emotional problems. These pupils are extremely sensitive and their feelings are easily hurt. They need to be given time to work without strain and encouragement to keep on working when the going is hard. Above all, they need consideration, sympathy, and love.

Whenever a counselor, after consultation with the first year algebra teacher and the homeroom teacher, believes that a pupil should not take plane geometry, she tries to guide him into another course which she feels would be of more value to him. In spite of the advice of the guidance department, there are always some pupils, who are ambitious or have ambitious parents, who insist that they want to study plane geometry. For some years the principal and the mathematics teachers have wondered what to do about these pupils, for they sat in class learning little or no geometry and just bothering the other pupils and the teacher.

Some people say that these pupils should not take geometry; however, the Fifteenth Yearbook of the National Council of Teachers of Mathematics says,[1]

"Backward pupils should not be taught only arithmetic, through the mistaken belief that arithmetic is the simplest mathematics since it is the most elemen-

* Presented at the 28th Annual Meeting of the National Council of Teachers of Mathematics, Chicago, Illinois, April 15, 1950.

[1] *The Place of Mathematics in Secondary Education* (Fifteenth Yearbook of the National Council of Teachers of Mathematics [New York, 1940]), p. 142.

tary. Parts of algebra and geometry are simpler than parts of arithmetic; furthermore, they may be more useful and more broadening. Slow pupils should be taught the simple parts of different divisions of mathematics. In all the branches of mathematics that are taught there must be a preponderance of concrete material and experience, abstract parts being introduced slowly and with moderation. A variety of subject matter can be successfully taught if it is presented in concrete form, if it is properly motivated, and if time is given for its mastery."

In February, 1943, a plan was tried at Thomas Jefferson High School by which the pupils who did not "see the light" in plane geometry at the end of six weeks were put in a special class that started back at the beginning of the course and took three semesters to cover one year's work. It was designated Geometry A, B, and C to distinguish it from Geometry 1 and 2. This plan helped a few students, but was discontinued because it had the following disadvantages:

1. The pupils did not like it because it took an extra semester. They felt that they were receiving no credit for the last one—Geometry C.

2. The Geometry A and B classes had so many drop outs and failures that there were hardly enough pupils left for a class of Geometry C.

3. The pupils still did little or no homework because they did not know how to study.

4. The teachers, as well as the pupils, became discouraged and did not like to teach the class.

Since September, 1947, a new plan has been tried in place of Geometry A, B, and C. Instead of waiting for a pupil to fail geometry, the ones who are likely to have trouble with it are picked at the end of their course in first year algebra. The Iowa Geometry Aptitude Test and the New California Short-Form Test of Mental Maturity are given. The results of these tests and the pupil's past record in mathematics are considered in determining whether he should go into the special geometry class—Geometry K. (Incidentally K does not stand for anything in particular.)

In order to overcome some of the disadvantages of Geometry A, B, and C, the class runs two consecutive fifty-minute periods, with one period for teaching and class discussion and the other for supervised study as a usual procedure. This gives time for a more leisurely approach to the subject. Since the pupils have more time in class, less work has to be done at home; and by working under the guidance of the teacher, correct methods of attack and good study habits are established. Pupils are encouraged to work in a relaxed manner without emotional stress. Tests of normal length for one period are given, but the strain of taking tests is lessened because pupils may use both periods if they wish. The first two periods in the morning when the pupils are not tired has been found to be the best time for the class. It is very important for this type of class to run smoothly, for when the pupils are upset, learning is blocked. There is more rustle and noise in such a class than in a regular one; and the teacher must decide whether the disturbance is a normal one or whether there is trouble. These pupils simply cannot sit still for a long time, and they must be allowed some freedom to move around.

Instead of the traditional way of teaching plane geometry, the class is taught by the laboratory method. Each pupil is encouraged to have an experimental approach to the subject and to discover geometric principles for himself rather than having them imposed upon him by the textbook or the teacher. As far as possible each pupil has his own model and is led by a series of questions to discover what happens if certain conditions are true. The "if-then" relationship is established, for the pupil knows that what is fixed on his model tells him what is "given" and that what results as he moves some

pieces of the model tells him what he must prove. The pupil states his own definitions and propositions, crudely at first, but these statements are later polished by the class; however, the pupil understands the proposition even though he can not state it clearly, because he has discovered it for himself. Afterwards the pupil proves the proposition. In this way he learns the difference between the experimental approach—inductive reasoning, and the geometric proof—deductive reasoning.

The laboratory method was found to be more effective in teaching these pupils than the traditional method which had been used in Geometry A, B, and C. Again the Fifteenth Year book points out.[2]

"Slow pupils, like very young children, learn best through experiences with concrete things. They must handle, measure, count, draw, make models, construct graphs, go on expeditions, hear talks, see pictures of things, etc., until the quantitative or spatial characteristics or relationships to be taught are really understood. After that the teacher may lead very gradually to the abstract, returning frequently to the concrete to fix and illustrate the generalizations."

A pupil who is physically handicapped or one who lacks muscular coordination can manipulate a model more quickly than he can draw the figures. Such pupils go through a terrific struggle to draw figures which often are so distorted that they can not see the relationships that exist. Freed from their physical handicaps and allowed to work with correct figures on models, they find geometry is fun, and they do some unique work. A boy who was a polio cripple did a marvelous piece of work on the trisection of an angle.

Emotional instability can be as great a handicap as mental immaturity. Perhaps you will be surprised that a boy with an I.Q. of 140 was placed in this class. After studying geometry six weeks in a regular class, he asked if he could come in Geometry K. In a conference with him the teacher asked him what she could do to help him.

He replied, "Teach me to be fast. Teach me to be fast."

His counselor said that he was on the verge of a nervous breakdown because he had brooded so much over geometry. His grades were low because he could not finish a test before the bell rang. He would not do anything outside of school except sit with his geometry book open before him from the time he reached home until he went to bed. He would not play or eat and neglected his other lessons. It was the only subject that kept him off the honor roll the first month. He was really pitiful to see. Once he started manipulating the models with his hands, all the tenseness left him and he learned to relax. Geometry became a fascinating subject which before had been dark and mysterious. He was still slow about finishing a test, but the strain of taking tests was relieved by the double period. He became a well adjusted normal boy and not the mental case his brother had been five years earlier.

Let's see how the pupils liked the laboratory approach to plane geometry.

BILLY—"Figures on paper are confusing. One model 'is worth a 1000 words.' Statements are easier to remember if you have an experiment and a model."

ROY—"On tests if you think of the model, you can work the problem."

JERRY—"You can see by the model what can be true and what is not true."

JOHN—"The experiments help one over the tough spots without doing the work for him."

DON—"With a model you are able to see what the figure looks like; the different ways it can be."

CHARLES—"You can tear down the figure and put it together again."

BOB—"Experiments help you learn and remember statements."

[2] *Ibid.*, p. 138.

KNOW YOUR NATIONAL COUNCIL REPRESENTATIVES

By KENNETH E. BROWN

Chairman of State Representatives, Department of Mathematics, University of Tennessee, Knoxville, Tennessee

WEST VIRGINIA REPRESENTATIVE

MISS MAY L. WILT has just completed her 25th year in the Laboratory High School at West Virginia University. The years have not dimmed her enthusiasm for mathematics or her interest in boys and girls in the high school. Miss Wilt has contributed pertinent articles in mathematics education to THE MATHEMATICS TEACHER, *Fifth Yearbook of the National Council of Teachers of Mathematics*, and the *West Virginia State Journal of Education*. She is the chairman of a committee which is developing a teacher training program for West Virginia. During the last seven years the National Council membership in West Virginia has doubled and Miss Wilt expects it to more than double again in the next seven years.

PENNSYLVANIA REPRESENTATIVE

MISS JOY E. MAHACHEK, Head of the Department of Mathematics, State Teachers College, Indiana, Pennsylvania, has been the National Council Representative in Pennsylvania since 1947. During this period the membership has grown from 400 to 550 members. She recently stated, "Pennsylvania has the third largest membership of the states. We hope to outdistance New York and Illinois soon."

Miss Mahachek is a native of Iowa, where she received the B.A. degree. Since leaving Iowa, she has been awarded both the master's and doctor's degrees. She has shown a great interest in mathematics education both as a supervising teacher and as a contributor to the Seventeenth Yearbook.

MAY L. WILT

JOY E. MAHACHEK

THE PRESIDENT'S PAGE

WHAT IS the National Council doing about cooperating with other national organizations?

In our new status as a Department of the National Education Association, we are in an ideal position to cooperate with any of the thirty other departments, seventeen of which maintain headquarters offices in the NEA building. Also, we have available the services of the fifteen NEA Divisions, among which are the Division of Accounts, Audio-Visual Instructional Service, Business, Legislative and Federal Relations, Publications, Records, and Research. While these opportunities of cooperation are open to us and much is to be realized from the development of our relationships with these organizations, we do not wish to limit our cooperative efforts to these alone. There are other national organizations, not within the NEA family, with whom we have worked in the past and with whom we expect to continue to cooperate.

For years the National Council and the Mathematical Association of America have worked together on major projects. One of the most important contributions to the field of mathematics in recent times is the final report of the Joint Commission of the Association and the Council which appeared as the Fifteenth Yearbook of the National Council of Teachers of Mathematics under the title of "The Place of Mathematics in Secondary Education." Since the cooperative work carried on in the past by the Association and the Council has been most agreeable and productive, it should be continued.

Because of our desire and increased efforts to give all help possible to the elementary teacher of mathematics, for the past three years we have been sending representatives to the Conference of Leaders in Elementary Education, an annual conference arranged and conducted by the Office of Education of the Federal Security Agency. Three years ago, Martha Hildebrandt, Proviso Township High School, Maywood, Illinois, represented the Council at the conference. Two years ago, Agnes Herbert, Clifton Park Junior High School, Baltimore, Maryland, was our representative. This past year we sent two representatives, Miss Herbert again and Mary C. Rogers, Roosevelt Junior High School, Westfield, New Jersey. More than forty national organizations participated in the fourth annual conference held in Washington last May.

Our representatives at these conferences have advised continuance of our cooperation with The Leaders in Elementary Education. They feel that the National Council is not only making valuable contacts with these other participating national organizations, but, at the same time, we are placing ourselves in a position to be of service to the U. S. Office of Education in its efforts to help the teaching situation in the elementary schools. Our contacts and relationship with the Office of Education promise to be mutually helpful. Particularly, and at the moment, this office can be of help to us in publishing leaflets and pamphlets on teaching aids and distributing these to the teachers of mathematics.

Incidentally, our participation in these conferences has emphasized again the need for a specialist in mathematics in the Office of Education. There are specialists in foreign languages, science, health, etc., but none in mathematics. This has been a matter of concern to the Council and we are taking advantage of every opportunity to urge that a specialist in mathematics be placed in the Office of Education. We are advised that this will be done as soon as a sufficient appropriation is granted by Congress to make it possible, and we are told that interested organizations will be given an opportunity to nominate candidates. We are keeping in close touch with developments along this line.

H. W. CHARLESWORTH, *President*

AFFILIATED GROUPS AND THE FUTURE OF THE NATIONAL COUNCIL

JOHN R. MAYOR, *Chairman*
Committee on Affiliated Groups

THE National Council of Teachers of Mathematics with more than 7000 members is already one of the largest professional organizations of teachers in any subject matter area. If the Council is to carry out its function to improve mathematics teaching at all levels of instruction everywhere, it not only should have a much larger membership but it will also have to constantly improve its services and make sure that its members and mathematics teachers everywhere make use of these services.

The Affiliated Groups can play an exceedingly important role in promoting the growth of the Council and in making available to teachers the services of the Council. In so doing, they will, of course, also be serving their own best interests. As an organization becomes larger it becomes much more necessary for the organization to work through smaller Affiliated Groups both to maintain the strength of the organization and to insure that the organization keeps in close touch with the needs of its individual members and others whom its services should benefit.

A major purpose of the Delegate Assembly is to determine how the National Council can most effectively work through the Affiliated Groups. The Affiliated Groups can greatly assist in building up the membership of the Council. The Affiliated Groups can make certain that the services of the Council reach their members and other teachers in their areas. The Affiliated Groups can have a real responsibility in maintaining the services of the Council and in determining the kinds of services that the Council should offer. The Second Delegate Assembly this month in Pittsburgh should find means for the Affiliated Groups to meet these opportunities.

The Delegate Assembly is a forum which makes recommendations to the Board of Directors concerning activities and policies of the Council. It may be that as the Council grows in effectiveness and membership that the Delegate Assembly will be called upon to assume greater responsibilities in the constitutional organization of the Council. At present the Delegate Assembly is not representative of the membership of the Council in the true democratic sense, since many members of the Council are not also members of an Affiliated Group, and, of course, the members of the Affiliated Groups are by no means all members of the National Council. An optimistic estimate would be that 40% of the members of Affiliated Groups are also members of the National Council. But even though the Delegate Assembly has no "legislative" responsibilities in the National Council, it can be of major importance and influence in the total program of the National Council through its recommendations to the Board of Directors and the Business Meetings and through its planning for the Affiliated Groups and National Council relationships.

Too, it must be kept in mind that the Delegate Assembly represents only a small part of the relationships between the Affiliated Groups and the National Council. It is given special emphasis in this Report because the Delegate Assembly is the point of emphasis for this month in the total Affiliated Groups program.

Goal for 1951

While there are now Affiliated Groups with which teachers in 34 states and the

province of Ontario may be actively associated, there are still areas with no state or local mathematics teachers group affiliated with the National Council. This means that mathematics teachers in these areas do not have as great an opportunity for participation in the total program of the National Council as teachers in other cities and states and it means that the National Council cannot as effectively carry its services to these areas.

Examination of the list of officers of the Affiliated Groups in the December number of THE MATHEMATICS TEACHER will show which areas are not yet affiliated with the Council. A letter from you to a friend in one of these states may get the movement started. Let's all help!

The Louisiana-Mississippi Branch of the National Council and the Association of Teachers of Mathematics in New England are two of the most active Affiliated Groups. Teachers in these areas have found it advantageous to cross state lines for area organization. This may also be the best plan in other parts of the country.

Pennsylvania Organizes

A copy of a Constitution for the Pennsylvania Council of Teachers of Mathematics was mailed to nine Convention Districts in time for their fall meetings. Each District thus had the opportunity to discuss it and to elect a delegate to the state-wide convention at Harrisburg after Christmas. At this time the state organization was formally approved and plans were made for affiliation with the National Council. Pennsylvania teachers deserve special praise for the energy with which they have carried on state organization plans while also planning for the big 1951 Annual Meeting of the Council in Pittsburgh.

A First for California, Colorado, and Oklahoma City

The California Mathematics Council, the Colorado Council of Teachers of Mathematics, and the Oklahoma City Mathematics Council were the first three Affiliated Groups to complete their annual renewal of affiliation for 1950–51.

Coming soon after these renewals were the official forms and fees from the Indiana Council of Teachers of Mathematics, Iowa Association of Mathematics Teachers, Association of Mathematics Teachers of New Jersey, Mathematics Section of the Texas State Teachers Association.

Association of Mathematics Teachers of New York State, Spring Meeting, May 12, 1951, Syracuse University, Syracuse, N.Y.

Just a year ago an interested group of mathematics teachers in New York State began plans for an organization of mathematics teachers within the state. Their purpose in forming such an organization is to unify the study of problems in the teaching of mathematics. A preliminary meeting, December 2, 1950, in Syracuse, N. Y., was attended by representatives of 35 New York counties. There, plans were made for an all-day meeting in the spring of 1951.

The temporary chairman, Miss Alice Reeve, 37 Clinton Avenue, Rockville Centre, N. Y., announces that plans are complete for a meeting on Saturday, May 12, at Syracuse University, Syracuse, N. Y. There will be general speakers as well as sectional meetings for groups ranging from college teachers of mathematics to elementary school teachers. Every one interested is invited to attend. For details concerning the program and the organization write to Miss Alice Reeve at the above address.

At this spring meeting the suggested name of ASSOCIATION OF MATHEMATICS TEACHERS OF NEW YORK STATE will be voted upon. A constitution will be presented for consideration and a permanent slate of officers will be elected.

MEETINGS, INSTITUTES, CONFERENCES

Annual Summer Meeting, National Council of Teachers of Mathematics, St. Olaf College, Northfield, Minnesota, August 20–23, 1951. Outstanding teachers and nationally known leaders in the field of mathematics will conduct meetings and present demonstrations at all levels of mathematics. The meetings will include general sessions, sectional meetings, laboratory sessions, discussion periods, and study groups. These study groups will be under the direction of prominent teachers and leaders for a continuing three-day session and will have limited enrollments. The latest in audio-visual aids will be exhibited: films, filmstrips, and models. Anyone having a model that has proved useful in the teaching of mathematics is asked to exhibit the model at this conference. Write to Mr. Emil Berger, Monroe High School, St. Paul, Minnesota, for information if you wish to exhibit material. It is hoped that models from all sections of the United States and Canada will enhance the exhibit of teaching aids.

The meetings will take place on the St. Olaf College Campus, forty miles south of Minneapolis and St. Paul on Federal Highway 65 (State Highway 218). The Jefferson Bus Lines and the Rock Island and Milwaukee Railroads provide excellent transportation facilities for north and southbound travel and make good connections with east and westbound travel. College dormitories will provide adequate accommodations with charges of $1.50 per day per person in double rooms and $2.00 per day per person in single rooms. All linen and towels will be furnished by the dormitory. All rooms will be available from Sunday, August 19 through Wednesday night, August 22.

Meals may be obtained at the college cafeteria with an average cost of $1.30 for breakfast and lunch. Special plans have been made for each evening as follows: Monday a picnic ($1.00), Tuesday a Norwegian Dinner ($2.00), and Wednesday a banquet ($2.50).

Reservations for rooms and registration for study groups should be made as soon as possible. Send reservations or requests for information and registration forms to Professor C. S. Carlson, St. Olaf College, Northfield, Minnesota. A registration and reservation form will appear in the May issue of THE MATHEMATICS TEACHER.

Facilities for recreation on the campus include a gold course, swimming pool, and tennis courts. There will be an evening trip to the observatory at Carleton College, folk dancing, and excursions. Northfield is the "southern gateway" to the magnificent northern Minnesota resort area with its numerous lakes and the famed North Shore Drive along beautiful Lake Superior. Plan a vacation trip to the "Land of 10,000 Lakes," and invite the mathematics teachers in your school or in other schools to meet you and your friends in Northfield at the end of August.

University of California at Los Angeles, Institute for Teachers of Mathematics, July 9–20, 1951. This Institute is sponsored by the Department of Mathematics and will be directed by Professor W. W. Rankin of Duke University. Two lectures daily will be given by outstanding men from business, industry, and teaching. The following study groups will be conducted daily by leaders of recognized ability chosen from teachers in the various fields of mathematical instruction.

1. Aids in the teaching of geometry and algebra
2. The development and coordination of basic ideas in mathematics
3. Teaching for understanding in arithmetic
4. Teaching applications of mathematics as used in industry and other fields
5. Use of mathematical instruments
6. Modern trends in the organization of secondary curricula in mathematics

Certificates of attendance will be issued to those registered in the Institute. It will also be possible for registrants to earn two units of college credit by attending lectures and participating in a sufficient number of the study groups.

Application blanks for the Institute and further details may be obtained by writing to Clifford Bell, Mathematics Department, University of California, Los Angeles 24, California.

Second Annual Meeting, Michigan Conference of Mathematics Teachers, St. Mary's Camp (Battle Creek), Friday to Sunday, May 11–13, 1951. The program of the conference is being planned to provide stimulating professional activities for mathematics teachers at all grade levels from the elementary school to the junior college. Enough time will be set aside to allow participants to take advantage of the excellent recreational opportunities which the camp affords, and to become well acquainted with their colleagues from other schools. The following program has been announced by the program committee: Friday—1:00 to 6:00, Registration; 4:30–5:30, Coffee Hour; 6:45–7:45, Dinner; 8:00–8:30, Short general meeting, address of welcome and formal opening of the conference; 9:30–(?), "Get-Acquainted" Hour, games and conversation, with coffee and "snacks" at 10:00. Saturday—8:00–8:30, Breakfast; 9:00–9:30, General Meeting; 9:30–10:30, Meetings of discussion groups; 10:30–11:30, Meetings of discussion groups; 12:00–1:00, Luncheon; 2:30–3:30, Meetings of discussion groups; 3:30–4:30, Meetings of discussion groups; 4:30–6:00, Recreation period; 6:00–7:00, Dinner; 7:30–8:30, General meeting of the conference with Mr. Walter H. Carnahan of Purdue University speaking; 8:30–(?), Recreation period, with coffee to 10:00. Sunday—8:00–8:30, Breakfast; 9:00–10:00, Meetings of

discussion groups; 10:00–11:00, General meeting of the conference; 12:00–1:00, Dinner.

The program committee is planning a varied array of topics with competent leaders for the small discussion groups, and is arranging also to provide observation, instruction, and practice in constructing multi-sensory teaching aids.

The cost of attending the conference is very moderate. The total cost to each individual will probably be twelve dollars which will include dormitory sleeping accommodations for Friday and Saturday nights and all meals beginning with dinner Friday evening and ending with dinner Sunday noon. This charge also includes the nightly coffee and "snacks" and the registration fee of one dollar. The camp can accommodate only about 150 persons. Reservations should be made early and each must be accompanied by a two-dollar deposit, which will be deducted later from the total charge. Requests for reservations should be sent to one of the following persons: Donald Worth, Lincoln Junior High School, Kalamazoo, Michigan; or William H. Nault, W. K. Kellogg Junior High School, Battle Creek, Michigan.

The Association of Teachers of Mathematics in New England, Institute for Teachers of Mathematics, Connecticut College, New London, Connecticut, August 23–29, 1951. The Association is happy to announce early plans for its third seven-day summer Institute. The response to the 1949 Institute at Wellesley College and the 1950 Institute at Tufts College has been most enthusiastic.

The Institute affords teachers of mathematics opportunity to discuss their subject with other men and women in the field, renew contact with sources of inspiration and enthusiasm, strengthen their confidence in the methods they are using, and enjoy pleasant association in an atmosphere of congeniality and good fellowship. Teachers in elementary schools, secondary schools, and colleges will all find features of interest at their particular level. You are invited to attend and have a part in making the Institute a great success.

The program will include outstanding speakers on topics in pure and applied mathematics, discussion groups on teaching methods and curriculum trends under the leadership of recognized teachers, a mathematics laboratory and workshop where models and other visual aids are constructed, and exhibits of latest texts and a variety of teaching materials. There will be abundant opportunity for "shop-talk" and a program of recreational features is being arranged for your enjoyment.

Connecticut College is ideally situated for the Institute, both in location and in facilities. The historic whaling port of New London is almost exactly half way between New York and Boston, on U. S. Route 1 and served by the Shore Line of the New York, New Haven, and Hartford Railway. The spacious campus is located on a hilltop, with views of Long Island Sound and the Thames River. The College Arboretum with its bridle paths and small lake, the nearby Ocean Beach Park, the miles of shore line and various state parks, all within easy driving distance, offer a wealth of recreational facilities. Teachers from a distance may combine a delightful New England vacation with a stimulating and enriching professional experience.

A tentative list of lectures and discussion groups follows. A pamphlet containing the complete program will be available in April.

Lectures

1. "The Quantitative Study of Interaction in Small Groups" Fred L. Strodtbeck, Assistant Professor of Sociology, Yale University, New Haven, Connecticut
2. "Cartesian Instead of Euclidean Geometry" Julia Wells Bower, Chairman, Department of Mathematics, Connecticut College, New London, Connecticut
3. "Models and Methods" Robert C. Yates, Professor of Mathematics, United States Military Academy, West Point, New York
4. "The Foundations of Arithmetic" H. W. Brinkmann, Professor of Mathematics, Swarthmore College, Swarthmore, Pennsylvania
5. "Map Projections" A. Day Bradley, Assistant Professor of Mathematics, Hunter College of the City of New York
6. "Graphics: The Neglected Mathematical World" John T. Rule, Head, Section of Graphics, Massachusetts Institute of Technology, Cambridge, Massachusetts
7. "Galaxies" (Illustrated) Harlow Shapley, Professor of Astronomy, Harvard University, Cambridge, Massachusetts
8. "Slides and Demonstrations in Connection with Geometry," H. von Baravalle, Professor of Mathematics, Adelphi College, Garden City, New York
9. Banquet Speaker: James R. Killian, President, Massachusetts Institute of Technology, Cambridge, Massachusetts

Discussion Groups

I. A. "The Junior-Senior High School Laboratory"
Leaders: Allene Archer, Thomas Jefferson High School, Richmond, Virginia, and Gladys Schuder, Lane High School, Charlottesville, Virginia

B. "How I Teach
i. Approximation
ii. Proof
iii. The Function Concept
iv. For Appreciation"
Leaders to be announced.

C. "Calculus for Teachers of Twelfth Grade Mathematics"
Leader: Edward M. Pease, Chairman, Department of Mathematics, Rhode Island State College, Kingston, Rhode Island

D. "Contributions to Enliven the Teaching of Mathematics in College"
Leader: H. von Baravalle, Professor of Mathematics, Adelphi College, Garden City, New York

II. A. "Arithmetic and Junior High School Mathematics"
Leader: Ben A. Sueltz, Professor of Mathematics, State Teachers College, Cortland, New York

B. "The Senior High School Laboratory"
Leader: George W. Kays, Instructor in Mathematics, State Teachers College, Montclair, New Jersey

C. "Modern Mathematics after the Tenth Grade"
Leader: Eugene P. Northrop, Chairman, College Mathematics Staff, and Associate Dean of the College, University of Chicago, Chicago, Illinois

III. A. "The Status of Mathematical Concepts and Skills at the End of the Sixth Grade"
Leader: Kenneth Fuller, Professor of Mathematics, Teachers College, New Britain, Connecticut

B. "Mathematics Clubs and Recreations"
Leader: E. H. C. Hildebrandt, Associate Professor of Mathematics, Northwestern University, Evanston, Illinois, and Editor of THE MATHEMATICS TEACHER

C. "The Correlation of Algebra and Geometry in Grades Ten and Eleven"
Leader: Virgil S. Mallory, Chairman, Department of Mathematics, State Teachers College, Montclair, New Jersey

D. "The Place of Postulates in College Freshman Courses"
Leader: Albert A. Bennett, Professor of Mathematics, Brown University, Providence, Rhode Island

The only fee for the Institute will be $10.00 for registration and a $1.50 laboratory charge for materials for those who join that group. Room and board in attractive modern college dormitories will be available at reasonable rates. To be sure of accommodations, reservations should be made before June 1. Commuters will also be welcome.

For any additional information, write to Charles H. Mergendahl, Newton High School, Newtonville, Massachusetts.

Ohio Council of Teachers of Mathematics, April 20–21, 1951. In November, 1950 a group of mathematics teachers met in Columbus, Ohio, to organize the Ohio Council of Teachers of Mathematics. The first general meeting will be held at the University High School, College of Education, Ohio State University, on April 20th and 21st. The topic is *Functional Mathematics.* Teachers of mathematics are cordially invited to attend. Notices and complete programs were sent the first of March to all on our mailing list. The secretary will supply others on request.

Temporary officers are: President, Herschel E. Grime, Cleveland; Vice-President, Harold L. Lee, Athens; Secretary, Ona Kraft, Cleveland; Treasurer, Oscar F. Schaaf, Columbus. H. C. Christofferson is chairman of the Program Committee, and Harold P. Fawcett of Arrangements.

The meetings begin with a dinner, April 20th at 6:30, with greetings by Dean Donald P. Cottrell, College of Education, and *Audio-Visual Aids in Teaching Mathematics,* by Edgar Dale, College of Education. At 10:00 A.M. on the 21st, *Mathematics as It Functions in Agriculture,* will be discussed by Chester S. Hutchinson, Assistant Dean, College of Agriculture. At 12:15, at a lunch and business meeting, a constitution will be adopted and officers elected. At 2:00, *Mathematics as It Functions in Industry,* will be presented by Charles E. MacQuigg, Dean, College of Engineering, and Salvatore M. Marco, Professor of Mechanical Engineering. Discussion groups are arranged for each session.

For further particulars address the secretary, Miss Ona Kraft, 11328 Euclid Avenue, Cleveland 6, Ohio.

Illinois State Normal University, Conference on the Teaching of Elementary and Secondary School Mathematics, Saturday, April 14, 1951, 9:00 A.M.–3:00 P.M. Speakers: Elementary Section, H. G. Wheat, West Virginia University; Secondary Section, E. H. C. Hildebrandt, Northwestern University. Sectional meetings will be held for both elementary and secondary teachers. For further information write T. E. Rine, Illinois State Normal University, Normal, Illinois.

Illinois Council of Teachers of Mathematics, Sectional Meeting, Saturday, April 28, 1951. This Conference is being held at Western Illinois State College and is sponsored jointly by the college and the Public Schools of Macomb. Principal speakers will be Miss Mary Potter, Supervisor of Mathematics, Racine, Wisconsin and Dr. H. Vernon Price of the University of Iowa. For further information write H. G. Ayre, Western Illinois State College, Macomb, Illinois.

North Western Wisconsin Mathematics Conference, April 7, 1951. This meeting will be held at the Eau Claire State Teachers College, with Dr. H. P. Fawcett of Ohio State University speaking on "Significant Problems in the Teaching of Geometry" and "The Interplay of Induction and Deduction in the Teaching of Geometry." Registration from 9:00–9:30 A.M. Registration fee: $1.00 (without luncheon), $1.50 (with luncheon). For further information write Lawrence Wahlstrom, State Teachers College, Eau Claire, Wisconsin.

MATHEMATICAL MISCELLANEA

Edited by PHILLIP S. JONES
University of Michigan, Ann Arbor, Michigan

19. Solutions of the Quadratic Equation.

Our *Miscellanea 5. A Geometric Solution of the Quadratic Equation*[1] brought us several interesting pieces of mail.

Professor Carl B. Boyer of Brooklyn College wrote to call our attention to Euclid's geometrical solutions of the quadratic. These solutions are to be found in Book II, Propositions 5, 6, 11 and Book VI, 27, 28, 29 of Euclid's *Elements*. The best known of these is II, 11 which solves $x^2+ax-a^2=0$ (a regarded as positive). This is the algebraic form of the division of a line in "extreme and mean ratio" and hence is the "golden section" which yields the "golden ratio" named the "divine proportion" by Pacioli (1509) and utilized in recent discussions of "dynamic symmetry." This problem is discussed in many places. Perhaps the most recent is an article by Carnahan[2] which is titled *Geometric Solutions of Quadratic Equations*.

Problems 85, 86 of Euclid's *Data* are also geometric equivalents of the solution of quadratics.

At this point, it should be noted that John Wallis' third solution which we gave in *Miscellanea 5*, figure 5, page 280 is, essentially identical with that given by René Descartes in *La Géométrie. Livre Premier* to be found on page 302 of his *Discours de la méthode pour bien conduire sa raison et chercher la vérité dans les sciences* (Leyden: 1637). Also, it would take only a slight modification of Wallis' first method (our figure 3, page 279, *loc. cit.*) to make it identical with the solution given by Descartes on page 303 of his work.

A letter from Willian J. Hazard of the Department of Engineering Mathematics of the University of Colorado includes the following list of 18 ways to solve $ax^2+bx+c=0$ taken from an article which he published in January 1924 in the *Colorado Engineer:*

1. By factoring by inspection.
2. By factoring after a substitution, $z=ax$, which leads to $z^2+bz+ac=0$.
3. By factoring in pairs by splitting bx into two terms.
4. By completing the square when a is 1 and b is even.
5. By completing the square as usual after dividing through by a.
6. By completing the square by the Hindu method ("the pulverizer"), i.e. by multiplying through by $4a$ and adding b^2 to both sides.
7. By completing square as given, adding $b^2/4a$.
8. By the formula.
9. By trigonometric methods (see Wentworth-Smith, *Plane Trigonometry*).
10. By slide rule (see Joseph Lipka, *Graphical and Mechanical Computation*. John Wiley and Sons, Inc. [1918], p. 11 ff.)
11. By graphing for real roots. (All modern textbooks.)
12. By graph, extended for complex roots. (See: Howard F. Fehr, "Graphical Representation of Complex Roots," *Multi-Sensory Aids in the Teaching of Mathematics, Eighteenth Yearbook of the National Council of Teachers of Mathematics* [1945] pp. 130–138. George A. Yanosik, "Graphical Solutions for Complex Roots of Quadratics, Cubics, and Quartics," *National Mathematics Magazine*, 17 [Jan. 1943], pp. 147–150.)

[1] THE MATHEMATICS TEACHER, vol. 43 (Oct. 1950), pp. 279–280.

[2] Walter H. Carnahan, "Geometric Solutions of Quadratic Equations," *School Science and Mathematics*, 47 (Nov. 1947), pp. 687–692.

13. Real roots by Lill circle. (d'Ocagne, *Calcul graphique et nomographie*, from which L. E. Dickson got his reference to it in his *Elementary Theory of Equations*.) (Also see J. W. A. Young's *Monographs on, Topics in Modern Mathematics* "Constructions with Ruler and Compasses.")
14. By extension of the Lill circle to include complex roots.
15. Using the graph of $y=x^2$ and $y=-bx-c$ to find real roots. (Lipka, *op. cit.* p. 26, modifies and extends this solution; Schultze, *Graphic Algebra;* Hamilton and Kettle, *Graphs and Imaginaries*.)
16. By extending (15) to include complex roots (Hamilton and Kettle, Schultze).
17. By use of a table of quarter squares. This is a practical method of handling an equation having large constants, as we already have the table in print (Jones' *Mathematical Tables*).
18. By use of "Form Factors."

Professor Hazard adds that methods 12, 14, 17, 18 are original with himself, and that 13, 14 and 17 will be discussed in his book *Algebra Notes* to be published soon. He also suggests a 19th method:

"Make a template of transparent plastic, very slightly smaller and parallel to the unit parabola $y=x^2$, so that the pencil or ruling pen will draw the true curve. Scratch co-ordinates on it for accurate placing, and for any equation move the template to a position parallel to the Y axis and with its vertex at the point

$$\left(\frac{-b}{2a}, \frac{4ac-b^2}{4a}\right).$$

While this can't be called a different 'method' of solution, I think it should be included as a very handy and practical 'variation of approach' to the graphic solution of the quadratic in one unknown."

We hope that Professor Hazard and others will extend our list and include expositions of some of the less obvious or less easily available procedures.

Before leaving this topic, however, we should mention the reprint sent us by Professor H. D. Larsen of "Solutions of the Quadratic Equation" which appeared in the Spring 1950 issue of *The Pentagon*. Professor Larsen, editor of the journal, writes that he has some copies of the article which he will sell for thirty-five cents each. The article was written by Raymond H. Gillespie as an Honors Thesis in Mathematics while he was an undergraduate at Albion College, Albion, Michigan where Professor Larsen is also Chairman of the Mathematics Department. This thesis includes an historical survey, early solutions (including Viete's solution via the substitution of $x=u+z$ in $x^2+ax+b=0$ and the equating of the coefficient of u to 0), derivations of the formula (including that via determinants by Euler and Bezout), methods of factoring, graphical solutions, determination of imaginary roots, and solutions by trigonometric methods. Although the tabulation of procedures is not, of course, absolutely complete and the historical survey ignores entirely recent discoveries in regard to Babylonian achievements, nevertheless it is well done and contains much interesting material which would enrich both classroom teaching and club programs.

20. Notes on Werner Todd's Trisection

Professor Norman Anning handed us some notes on *Miscellanea 4. Trisecting Any Angle* (THE MATHEMATICS TEACHER, vol. 43 (Oct. 1950), pp. 278–279), with the admonition to destroy his message and then to keep away from trisections as we promised earlier. We can no more destroy them without some mention than he could resist the temptation to give them to us.

The curve which Mr. Todd constructed is one half of the Trisecant of Delanges[3]

[3] Described by P. Delanges in *La trisegnante nuova curva* (Verona: 1783) according to Gino Loria writing in *Enciclopedia delle Mathematiche*

a quartic curve of which a cartesian equation is $(y^2-4m^2)(x^2+y^2)+4m^2=0$.

Professor Anning further notes that this method reappeared in 1943 in the work of a William Joseph Frederick of Battle Creek, Michigan, who constructed the curve by a clever combination of compasses and a T-square. The year before this the same curve was used for this purpose by Harvey P. Sleeper (Harvard, 1942, Eliot House). Sleeper's mechanism is to be found on the endpaper of an early printing of R. C. Yates, *The Trisection Problem* (1942) an excellent reference on this topic which is still available from the author at the United States Military Academy, West Point, New York.

21. Inequalities, the Quadratic, and the "Ambiguous Case."

The "ambiguous case" in the solution of oblique triangles is, of course, the case in which two sides and an angle opposite one of them are given. The various situations in which 0, 1, or 2 solutions occur are usually analyzed from two drawings, one of the acute- and the other of the obtuse-angled case. This is desirable visualization.

Analytically, in the acute angled case where sides a and b and angle A are given, the altitute $h_c=b \sin A$ is computed and the nature of the solutions determined by comparing h_c with a. This is convenient since $b \sin A$ is simultaneously the natural first step in determining B via the Sine Law. If A is obtuse, $a>b$ is necessary and sufficient to guarantee a unique solution.

Another approach which may be used as an exercise or extra credit project for the pedagogical purpose of making an interesting display of the interconnections between this problem, the quadratic discriminant, and conditional inequalities is as follows.

As above consider a, b, and A to be given. Then $a^2=b^2+c^2-2bc \cos A$ may be considered as a quadratic equation in c. From this

$$c=b \cos A \pm \sqrt{b^2 \cos^2 A-b^2+a^2}.$$

Since negative or zero values for a, b, c have no geometric significance, we see that (1) if $\cos A$ is positive (i.e. if A is acute) there may be 0, 1, or 2 solutions while (2) if $\cos A$ is negative (i.e. if A is obtuse) there will be one and only one solution if and only if $b^2 \cos^2 A-b^2+a^2 > b^2 \cos^2 A$ which condition simplifies to merely the requirement that $a>b$.

In situation (1) there will be no solution if $b^2 \cos^2 A-b^2+a^2<0$. But $b^2 \cos^2 A-b^2 = -(b \sin A)^2$. Hence the original condition reduces to the requirement that $b \sin A>a$ or that $h_c>a$. From this it readily follows that there will be a single unique solution if $b \sin A=h_c=a$, and if $h_c<a$ there will be two solutions as long as $b \cos A>\sqrt{b^2 \cos^2 A-b^2+a^2}$. There will be only a single solution however if $h_c<a$ and $b \cos A \leqq \sqrt{b^2 \cos^2 A-b^2+a^2}$. This latter condition is easily seen to be equivalent to $b \leqq a$.

Elementari e Complementi, volume II, part II, (Milano: 1938), p. 394 ff. See also Heinrich Wieleitner, *Spezielle Ebene Kurven* (Leipzig: 1908), pp. 79–82.

Second Annual Mathematics Institute, Louisiana State University, June 24-30, 1951

There will be study groups for elementary, junior high and senior high school teachers on arithmetic and high school mathematics. In addition there will be study groups on enrichment materials and the slow learner. Leaders for these groups are outstanding specialists from various parts of the country. Also there are to be general sessions. Some of the best talent in the field of the teaching of mathematics and other fields of interest will speak at these sessions. A new feature will be a laboratory where models will be demonstrated and made. Rooms and meals will be available on the campus at reasonable rates. Everything possible is being done to insure that everyone will have a grand time together and a profitable experience. Plan now to attend this second L.S.U. Mathematics Institute. A printed program will be available after March 1st. For further information, write Houston T. Karnes, Department of Mathematics, Louisiana State University, Baton Rouge 3, Louisiana.

NOTES ON THE HISTORY OF MATHEMATICS

Edited by VERA SANFORD
State Teachers College, Oneonta, New York

THE PROBLEM OF THE LION IN THE WELL

IT IS proposed to treat under this head the problems dealing with the time needed to go a certain distance at a constant rate when this progress is hampered by a constant retrograde motion.

The problem seems to be of Hindu origin, and as will appear from the following examples, the situation was elaborated in various ways. In the work of Mahāvīra (*c.* 850), there are at least two examples of this problem:

"In the course of 3/7 of a day, a ship goes over 1/5 of a krosá in the ocean; being opposed by the wind she goes back during the same time 1/9 of a krosá. Give out, O you who have powerful arms in crossing over the ocean of numbers well, in what time the ship will have gone over 99 2/5 krosá?"

"A powerful unvanquished excellent black snake which is 32 hastas in length, enters into a hole at the rate of 7 1/2 angulas in 5/14 of a day; and in the course of 1/4 of a day, its tail grows by 2 3/4 of an angula. O ornament of arithmeticians, tell me by what time this serpent enters fully into the hole."

About three hundred and fifty years later, Fibonacci gave the problem this way:

"There is a lion in a well whose depth is 50 feet. He climbs 1/7 of a foot daily and slips back 1/9 of a foot. In how many days will he get out of the well?"

Fibonacci (1202) has a more complicated case in which two serpents are approaching each other from the top and bottom of a tower respectively. Each advances a certain distance each day and retreats a certain distance each day. It is asked how long it will be before they meet. Borghi (1484) applies the same idea to two brothers who are trying to pay off a debt. Each earns a certain amount a day and each has certain daily fixed expenses. When will the debt be paid?

A distinctly new form of the problem which appeared in the Middle Ages showed that the periods of progress and regression might alternate. It should be noted that this introduced a further complication for the progress of the last day is all gain. Fibonacci for example gives the case of a lion in a well who climbs up in the daytime and slips back at night. Similar problems were given by Italian and by German authors. An additional modification was to have two animals such as a hawk and a dove approaching each other from the top and the bottom of a tower. Calandri (1491) thought the problem worthy of a woodcut showing the serpent emerging from the well.

One of the most absurd variations occurs in Pacioli's *Summa* (1494):

"A mouse is at the top of a poplar tree 60 feet high and a cat is on the ground at the foot of the tree. The mouse descends 1/2 of a foot each day, and at night it turns back 1/6 of a foot. The cat climbs one foot a day and goes back 1/4 of a foot at night. The tree grows 1/4 of a foot each day between the cat and the mouse and shrinks 1/8 of a foot at night. In how many days will the cat reach the mouse, how many feet does the tree grow in this time, and how far does the cat climb?"

Buteo (1559) makes use of a situation which was characteristic of navigation in the Mediterranean:

"Two ships 20,000 stadia apart weighed anchor to set sail straight toward each other. It happened that the first one set

sail at daybreak with the North wind blowing. Toward evening, when it had gone 1,200 stadia, the North Wind fell and the Southwest Wind rose. At this time the other ship set sail and sailed 1,400 stadia during the night. The first ship, however, was driven back 700 stadia by the contrary wind, but with the morning North Wind it was driven ahead in the usual manner of outward sailing while the other went back 600 stadia. Thus alternately night and day, the ships were carried along by a favorable wind and then driven back by an unfavorable one. I ask how many stadia the ships sailed in all and when they met."

This problem situation remains in the puzzle of the frog jumping out of a well where the depth of the well is given, the amount the frog goes up each day and the amount he slides back each night, but the point of this case lies in seeing that the last day's progress is all gain, and the problem has found its place among mathematical recreations rather than among the more serious problems.

It is important to notice that the arithmetic side of the work in the puzzle is more simple than in the form given by Mahāvīra, but where the computational side has been simplified, the logical difficulties have been increased. If we consider this problem as allied with the familiar one of the man rowing with or against the current of a river, we have moved into a converse case where the time and distance are given and the two rates are to be found. The computation has been simplified and the algebraic interest has been increased. Here reference should be made to the various problems connected with air speed, ground speed and the velocity of the wind that appear in the various discussions of mathematics and aviation. In this connection, the problem which was real as applied to the navigation of two sailing ships in the work of Buteo, becomes a very real one today in computing the radius of action of an airplane—i.e., given the air speed and the direction and force of the wind, to compute the maximum distance an airplane can go in a given direction and return to its starting point.

References

The quotations from the work of Mahāvīra are taken from the *Ganita-Sāra-Sangraha of Mahāvīracārya* made by M. Rangacarya, Madras 1912.

The illustration from Calandri appears in the second volume of David Eugene Smith's *History of Mathematics* (1925), p. 540 and in Vera Sanford, *Short History of Mathematics* (1930), p. 208.

The text of the problems quoted here appears in Vera Sanford, the *History and Significance of Certain Standard Problems in Algebra*, New York, 1927, pp. 63–66.

Mathematics Workshop at Kent State University
Kent, Ohio, April 14, 1951

The program will begin at 9:30 with a talk by Lt. Col. Frank C. Mandell, U.S.A. on "Mathematics for the Draftee." Two class demonstrations will be held, one at 10:15 with a seventh grade class in geometry taught by Arthur Pheil of Kent State University High School and the second at 1:30 P.M. with a tenth grade geometry class taught by Emalou Brumfield of the Department of Mathematics at Kent State University. Five discussion groups will meet at 12:30 with topics and leaders as follows:

a) The Importance of Reading and Writing in the Study of Mathematics. Led by Gordon Bechtel, Strongsville High School
b) What's Wrong with our Present General Mathematics Course? Led by Olive Davis, Kenmore High School, Akron, Ohio and Ruth White of Breckville High School
c) Film Forum. Led by E. T. Stapleford, Department of Mathematics, Kent State University
d) A Consideration of the Gifted Children in Your Classes. Led by Dr. L. L. Lowenstein, Head, Department of Mathematics, Kent State University
e) The Problem of Teaching Locus. Led by Carl Lowry, Department of Mathematics, Kent State University

A talk on "Game Theory" by Foster Brooks of Kent State University at 2:30 and a Review Assembly at 3:15 will complete the day's program. Teachers are invited to bring mathematical materials or devices for the mathematics exhibit. For further information write Miss Emalou Brumfield, Department of Mathematics, Kent State University, Kent, Ohio.

APPLICATIONS

Edited by SHELDON S. MYERS
University School, Ohio State University, Columbus, Ohio

WE ARE trying hard to keep errors and ambiguities at a minimum in this department. Do not hesitate to write us if anything does not seem to be quite right. We would like to correct two items in C.Al. 1 Gr. 10–12 Arithmetic Series in the Home. The left arrow indicating the horizontal length of the steps should stop at a point vertically beneath the top of the steps. The formula $S=n(a+L/2)$ should read $S=n(a+L)/2$.

Ar. 7 Gr. 7–9 A SEVENTH GRADE APPLIES SOME MATHEMATICS TO THEIR OWN HEIGHTS AND WEIGHTS

Here is a guide sheet designed and used by the writer at the University School, Ohio State University.

Last week you were all weighed and measured for your core unit on "Understanding my Body." With these figures you will be able to answer many interesting questions. In order to help you answer these equations, the weights and heights for boys and girls are printed on the attached sheet without names. Let us all follow this plan today in studying your weights and heights:

1. First we shall all read silently these instruction sheets.

2. Next, as a group we shall add to the list of questions any other which interests you and which deals with the information.

3. Before any of us begins to write the answers to the questions, we shall discuss briefly how to go about answering each one. This is the time for you to ask questions. You should take notes on what is said at this time.

4. Your assignment will be to work out the answers to the questions.

Some Interesting Questions to Answer About Heights and Weights

1. If all the boys climbed onto one end of a giant teeter-totter and all the girls climbed onto the other end, which end would go up and which end would go down? Why?

2. There are 2000 pounds in each ton. How many tons does the seventh grade weigh?

3. If you lived in France, you would always give your weight in kilograms. Now if there are 2.2 pounds in every kilogram, how many kilograms would you weigh? (Each student knew his own weight and height.)

4. The metric system has been adopted in every country of the world except the United States and Great Britain. In all these countries, the heights of people are given in centimeters. If every inch of your height contains 2.54 centimeters, how many centimeters high are you?

5. Can you tell from the data whether the heaviest seventh graders eat the most? What plan would you follow to find out?

6. What do you think makes a person taller or heavier than usual?

7. What do we mean by the word "usual" in the above question? Does "usual" height mean the *middle* height or the *average* height?

8. Find the "usual" height and weight for all seventh graders and for boys and girls separately.

9. It is not easy to picture in one's mind how the weights and heights of a large group distribute themselves by looking at a page of numbers. A bar graph showing the number of students in each weight and height bracket helps greatly. Using the attached, seventh grade height and weight data, make bar graphs for height and weight. Be sure to label each axis properly and to give each graph an identifying title. (The group decided to use the following intervals for weight: 50–59, 60–69, etc. up to 129; and the following intervals for height: 55–56, 57–58, etc. up to 66.)

DEVICES FOR A MATHEMATICS LABORATORY

Edited by EMIL J. BERGER
Monroe High School, St. Paul, Minnesota

THIS section is being published as an avenue through which teachers of mathematics can share favorite learning aids. Readers are invited to send in descriptions and drawings of devices which they have found particularly helpful in their teaching experience. Send all communications concerning Devices for a Mathematics Laboratory to Emil J. Berger, Monroe High School, St. Paul, Minnesota.

THE SPHEROMETER*

THE radius of a sphere and the radius of curvature of a spherical mirror or optical lens can be found rather conveniently by using a spherometer. Descriptions of this instrument can be found in textbooks on physics.

It is the object of this article to suggest plans for construction of a homemade spherometer which high school students of mathematics will find both interesting and challenging. The principle of operation involved is about the same as that of commercial instruments. Properly completed and calibrated this inexpensive instrument will provide students with a nice application of two important mathematical principles; these are: (1) that an angle inscribed in a semi-circle is a right angle; and (2) that the altitude on the hypotenuse of a right triangle is the mean proportional between the segments of the hypotenuse.

Construction materials needed include a standard one-pound coffee can, an unsharpened pencil, and a block of wood 1″×3″×3″. The device is assembled in the following manner (See Figure 1.):

(1) Nail the block of wood to the inside of the cover of the can.
(2) Drill a hole at E through the center of the cover and through the block of wood. This hole should be just large enough to accommodate the pencil snugly.
(3) Remove the bottom of the can with a can opener. An opener with a rotary blade such as the "swing-away" type will leave the circular edge nice and smooth.
(4) Replace the cover on the can and push the pencil through the hole at E. The instrument is now complete except for calibration of the pencil.

An alternative method of making the device is to fasten the block of wood to the inside of the bottom of the can, drill a hole through the center of the bottom, invert the can, and insert the pencil from the outside through the bottom. With this method of construction the bottom of the can is not removed, and the cover is not

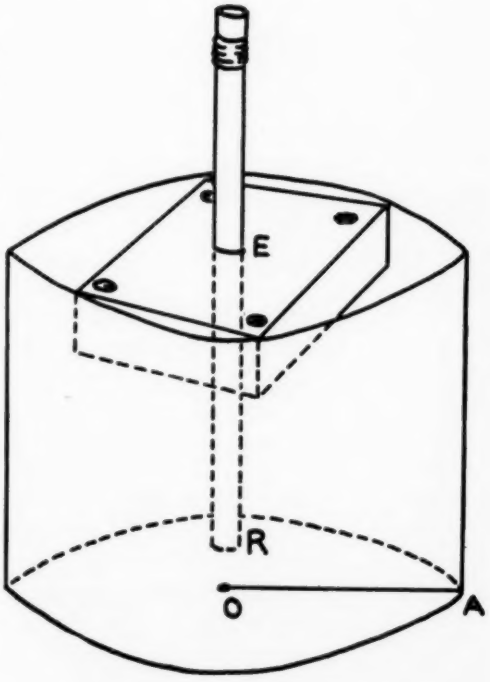

FIG. 1

* Plans for the construction of The Spherometer were developed in The Mathematics Laboratory, Monroe High School, St. Paul, Minnesota.

needed at all. Advantages of the first method are that it is easier to fasten the block of wood to the cover than to the inside of the bottom, and also the rim retained on the bottom after removing the tin is sturdier than the thin edge at the top of the can.

In order to calibrate the pencil—that is, to put marks on it so that various radii of curvature may be read directly—it is necessary to take into account several facts about the instrument which are features of its construction. The open end, or base, is a circle whose radius is OA. See Figure 1. For the usual coffee can the length of this radius is approximately 2.5″. If the circular base is placed on a plane surface and the pencil pushed completely down, the end point R will coincide with O. This position of the pencil may be recorded by making a notch in it at the point E where it enters the top of the can. Subsequently this first notch can be used as a reference point for recording other positions of R. Attention is called to the fact that when the notch just made coincides with the top of the can, or what is the same thing, when R coincides with O, the radius of curvature is infinite.

With the foregoing facts in mind let us proceed to determine the radius of an actual sphere; and, for the sake of concreteness, let this sphere be a basketball. Place the circular base of the instrument on the ball, and apply enough downward pressure to the top of the pencil to insure that the point R will rest on the surface of the ball. Observation will reveal that the original notch on the pencil has now moved up to the position E'. See Figure 2. Thus OR, which is the distance of R above the plane of the circular base, can be found by measuring EE'. Having determined the two measurements OA ($=2.5''$) and OR ($=EE'$), the radius of the ball may readily be computed. Let RS be the diameter of the ball; then triangle RAS is a right triangle, and OA is the altitude upon the hypotenuse. Thus OA is the mean proportional between the segments OR and OS, and the radius of the ball ($RS/2$) can be obtained from the following relations:

$$\frac{OS}{OA}=\frac{OA}{OR}$$

$$OS=\frac{\overline{OA}^2}{OR}$$

$$RS=OS+OR$$

$$\frac{RS}{2}=\text{radius of the ball.}$$

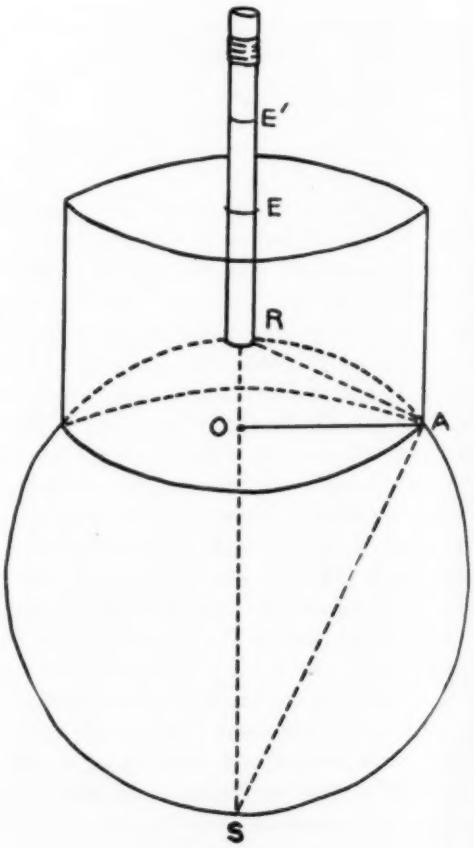

Fig. 2

By notching the pencil again at the level of E, and keeping a record of the length of EE' for the radius just computed, one operation in the process of calibration will have been completed. By repeating the above process with different sized spheres

various radii of curvature may be computed and recorded. Actually, however, the calibration for a desirable range of radii may be completed by measuring EE' for different positions of R, and simply doing the computation. Once the device has been calibrated it may be used to find the radius of curvature of some particular spherical surface simply by noting which notch appears at E and reading the computed radius corresponding to this notch from the recorded data.

Two items of interest claim attention at this point. The first is that it is not possible with this device to determine the lengths of radii which are less than 2.5″. The second is that while the discussion thus far has been concerned only with convex surfaces, the device can also be used to find radii of curvature of concave surfaces. To do so, however, the pencil must be calibrated in the direction above the original notch.

It hardly seems necessary to caution that measurements obtained with this device will be rather crude approximations; the materials and methods of construction employed are certainly not of the kind that should encourage much reliance on results if a high degree of accuracy is needed. The all important thing to remember is that the entire object of having students build a device of this kind is that their understanding of the principles involved will be enhanced.

A Device for Introducing the Number π

The usual approach to the introduction of the number π at the junior high school level places emphasis on two points: (1) that π is defined as the ratio of the circumference of any circle to its diameter; and (2) that the approximate value of the number is somewhere in the neighborhood of 3.1416 or $3\frac{1}{7}$. In early approaches the value $3\frac{1}{7}$ usually finds more favor among teachers. Undoubtedly every teacher of mathematics has some

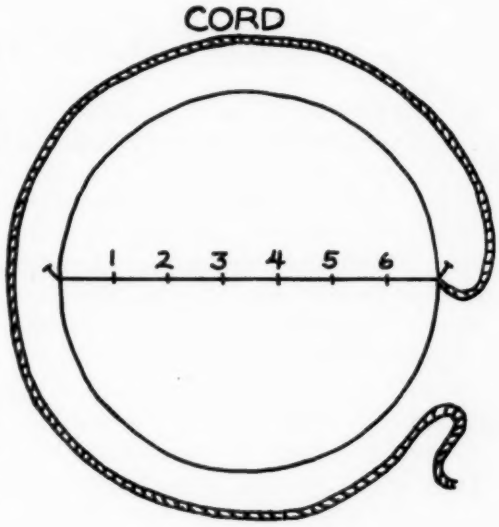

Fig. 3

technique of his own for giving meaning to the ratio definition of π and also some handy intuitive approaches for showing that a close approximation to the value of the number is $3\frac{1}{7}$. The device suggested here for developing these two points of emphasis was called to the attention of this department by Miss Ruth Knaus of Waseca High School, Waseca, Minnesota. It is a clever little teaching aid and much more tidy than the usual ones.

Materials needed to construct the device include a piece of plywood 7″ square, a piece of cord 22″ long, and two right-angled curtain hanger hooks. See Figure 3. Cut a circle of diameter 7″ from the plywood square, draw a diameter on its surface, and mark off seven one-inch divisions on it. Number the division marks along the diameter for easy reference. At the ends of the diameter screw the right-angled curtain hanger hooks into the sawed edge of the circle, and fasten the 22″ cord to one of these hooks. If wound around the sawed edge of the circle, the cord should measure about one circumference. When wrapped around the two hooks at the ends of the diameter the cord will measure $3\frac{1}{7}$ times the diameter.

MATHEMATICAL RECREATIONS

Edited by AARON BAKST
School of Education, New York University, New York 3, N. Y.

THE APPEAL of this Department to the readers of THE MATHEMATICS TEACHER for materials and problems which would enrich, enliven, and motivate the classroom teaching did not go unheeded. The number of letters and replies from the readers has intrigued the secretary of the Mathematics Department of the New York University School of Education, MISS JOAN GEASA, so much that she has made a contribution of her own. Here it is.

Two parts of a cross.
A circle complete.
Two semi-circles, and
A line down to meet.
Angle, triangle, standing
On two feet.
Two semi-circles
And a circle complete.

Don't peek. The answer is given at the end of this column. This problem should be very intriguing in a second-term geometry class.

And now, here is a problem which was sent in by Miss PAULINE WILSON of James Hillman Junior High School, Youngstown, Ohio. She writes: "I wrote this problem about three years ago and have used it with my beginning algebra class. I have continued using it (although I am conscious of its flaws) because the response is always gratifying. The pupils (and even some of their other teachers) take an interest in the solution. The pupils' reactions to the little mystery are far different from those to the problems about when and where two cars will pass."

THE CASE OF THE MISSING LAUNDRY WHO STOLE MRS. MURPHY'S WASHING?

Mrs. Murphy was naturally very angry when she learned that her washing had been stolen from her clothes line while she was away. She asked Detective Curlock Foams to find out who the thief was. Upon investigating the movements of all who had been in the neighborhood about that time, Detective Foams discovered the following facts.

There was but one road between Waterloo and Daleville (24 miles due north). Mrs. Murphy lived on this road just 15 miles north of Waterloo. At 10:00 a.m. a bus left Waterloo and arrived at Daleville at 10:40 a.m. without making any other stops. The same bus made a return trip at the same speed, leaving Daleville at 10:50 a.m. Passengers on the northbound bus remembered seeing the laundry on the line as they passed. The driver and some passengers on the southbound bus were certain that there was no washing on the line when they passed Mrs. Murphy's home.

At 9:30 A.M. Claude Bennington was known to have left his home 13 miles north of Waterloo. He walked north, past Mrs. Murphy's home, arriving at a service station, 18 miles from Waterloo, at 11:10 A.M. (Curlock Foams counted on a man walking at an average of 3 miles per hour.)

Claude's brother, Ralph, left the same house thirty minutes later, and he also walked north. When he was 17 miles north of Waterloo he was picked up by George Pumpernickle and was driven by him into Daleville.

Mr. Pumpernickle had left his home (4 miles north of Waterloo) at 11:10 A.M. and had driven at an even speed of 40 miles per hour all the way, stopping only a moment to pick up Ralph. He was certain that Ralph was carrying no packages of any kind.

At 10:00 A.M. Alex Martin left his

home (11 miles north of Waterloo) and walked to a garage (16 miles north of Waterloo), arriving there at the same time as did George Monroe.

George Monroe had travelled on his bicycle at the rate of 6 miles per hour from his home (10 miles north of Waterloo), having left home at 10:40 A.M.

Curlock Foams sat down at his desk, drew some lines on a piece of graph paper, picked up his hat, and went out to arrest the guilty person.

WHOM DID HE ARREST?

The answer to this problem will be found in the May issue of THE MATHEMATICS TEACHER. Please do not write to Miss Wilson. However, this problem does not require any special abilities other than those which are acquired in the study of simple graphical methods employed in any ninth year algebra.

Professor JULIUS SUMNER MILLER of Dillard University, New Orleans, La. submitted several very intriguing problems. We shall publish two of them now.

1. The product $2^5 \cdot 9^2$ was accidentally set up by the printer as 2592. Curiously enough there is no mistake. Is there another product like this?

2. On an examination a student copied.

$$\frac{5^2}{8}+\frac{3}{8} \quad \text{as} \quad \frac{3^2}{8}+\frac{5}{8}.$$

Curiously enough, their sums are the same. Is there another such pair?

The answer to the little ditty is TOBACCO. Every letter should be printed in capitals.

"*THIS* is what I've been dreading ever since school began . . . CALCULUS!"

THE SATURDAY EVENING POST

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TOPICS OF INTEREST TO MATHEMATICS TEACHERS

Edited by WILLIAM L. SCHAAF

Department of Education, Brooklyn College, Brooklyn, N. Y.

The Numerical Solution of Equations

To OBTAIN the solution of quadratic, cubic and quartic equations, a variety of algebraic methods are available. But in engineering and technological work it is frequently desirable to obtain approximate numerical values of all the roots not only of such equations, but also of higher degree polynomial equations as well as transcendental equations. Perhaps the best-known of several approximation methods is the scheme introduced by William George Horner in 1819, a method which is very efficient and relatively simple, although it entails considerable computation, and is limited to finding the real roots of a rational, integral algebraic equation. Inasmuch as Horner's method is quite familiar, it will not be described here.

Another well-known scheme is Newton's method, which, according to Cajori, is not really Newton's method at all, but a modification of it developed by Joseph Raphson (c. 1690); hence it is frequently called the Newton-Raphson method. Actually, the basic idea was anticipated by Vieta (1540–1603). To solve an equation $y=f(x)=0$ by Newton's method, we first find an approximation x_1 to the root r by graphical or other methods. If x_1 is chosen sufficiently close to r, we may then obtain a second and closer approximation x_2 by finding the point $(x_2,0)$ at which the tangent to the curve at the point (x_1,y_1) intersects the x-axis. The slope of this tangent is $y_1'=f'(x_1)$; hence

$$\frac{0-y_1}{x_2-x_1}=y_1',$$

or

$$x_2=x_1+\frac{-y_1}{y_1'}.$$

This is equivalent to using the first two terms (dropping all higher-degree terms) in Taylor's expansion of $f(x)$ about the point whose abscissa is x_1 and assuming that $y_2=0$. If $y_2=f(x_2)$ is not sufficiently small, we apply the method once more, compute a second correction $-y_2/y_2'$, and obtain a third approximation $x_3=x_2-y_2/y_2'$. The Newton-Raphson method can also be used to solve certain kinds of transcendental equations.

The method of false position (*regula falsi*) is essentially a method of inverse linear interpolation. It consists of making a fairly close estimate of a root, say r, then substituting $(r+h)$ in the equation, dropping the terms in h of higher degree than the first (since they are relatively small), and solving the resulting linear equation for h. The process is then repeated, using the approximation just found $(r+h)$ in place of r. In effect, the desired root r is inclosed in a nest of intervals whose end points are approaching r on at least one side. A modification of the method of false position is known as the method of chords.

The above four methods are all alike in that they consist of repetitive processes that make it possible to secure successively better and better approximations of a real root. However, the Newton-Raphson method differs from the other three in that an approximate value for the root is substituted in a formula, namely,

$$r_n=r_{n-1}-\frac{f(r_{n-1})}{f'(r_{n-1})},$$

in order to find a better approximation. The latter approximation is substituted *in the same formula* to find a still closer approximation, etc. In this respect the

Newton-Raphson method is a special case of the so-called method of iteration. In general, a method of iteration refers to any procedure whereby an approximation to a root is substituted in some arbitrarily selected function in order to yield a better approximation; the latter approximation is then substituted *in the same function* to yield a still better approximation, and so on. Various iterative methods use different functions for this purpose.

Still another scheme is Graeffe's method, known for some time, but not often mentioned in algebra textbooks. It was known to Dandelin (1794–1847), but not widely used in his day. It had been anticipated by Edward Waring (c. 1770). Brought into prominence by Carl Heinrich Graeffe (1799–1873), it was further developed by the astronomer Johann Franz Encke (1791–1865). The basis of Graeffe's method is a transformation called the "root-squaring" process, the details of which cannot be given here. The method is described in such well known treatises as Whittaker and Robinson, *Calculus of Observations* (London, 1924); Runge and König, *Numerisches Rechnen* (Berlin, 1924); Doherty and Keller, *Mathematics of Modern Engineering*, Book I (New York, 1936); Scarborough's *Numerical Mathematical Analysis* (Baltimore, 1930); and Uspensky's *Theory of Equations* (New York, 1948). A very illuminating discussion is also to be found in the paper by C. A. Hutchinson, listed below. The chief advantage of Graeffe's method is that it yields the imaginary roots as well as the real roots.

1. General Approximation Methods

Bailey, V. A. "Prodigious Calculation." *Australian Journal of Science*, 1941, 3: 78–80.

A method for approximating rationally to a root α of the equation

$$\phi(x) \equiv x^m - c = 0, \qquad c = \alpha^m > 0$$

using the function

$$f(x) = x \cdot \frac{(m-1)x^m + (m+1)c}{(m+1)x^m + (m-1)c}$$

in conjunction with a recursion formula.

———. "Solutions of Algebraic Equations by Means of Tables of Logarithms." *Philosophical Magazine*, 1934, s7, 18: 529–539.

Bateman, Harry. "Halley's Method for Solving Equations." *American Mathematical Monthly*, 1938, 45: 11–17.

Edmund Halley (1723) deserves the credit of having found approximation formulas which usually make the convergence of the iteration process extremely rapid when the first approximation is fairly good. His method forms the basis of modern iteration methods.

Bell, Clifford. "The Solution of Numerical Equations." *National Mathematics Magazine*, 1940, 14: 435–438.

Describes a practical method for finding approximate values on opposite sides of the root when applied to a proper interval, thus determining the maximum error in the approximation of an irrational root; the method makes use of the osculatory parabola drawn to the curve.

Bolz, H. A. "Higher Degree Equations Solved by Trial and Error." *Product Engineering*, 1943, 14: 217; 384–385.

Boon, F. C. "The Solution of Equations by the Use of Proportional Differences." *Mathematical Gazette*, 1937, 21: 182–187.

Discussion of an iterative process for the approximate solution of algebraic and transcendental equations.

Brock, J. E. "Note on the Method of Iteration." *Journal of Engineering Education*, 1949, 40: 263–264.

Buchner, P. "Zur Berechnung der Wurzeln höherer Gleichungen." *Zeitschrift für Mathematischen und Naturwissenschaftlichen Unterricht*, 1927, 58: 163 ff.

Cajori, Florian. "A History of the Arithmetical Methods of Approximation to the Roots of Numerical Equations of One Unknown Quantity." *Colorado College Publications, General Series*, Nos. 51, 52. 1910.

Coate, G. T. "On the Convergence of Newton's Method of Approximation." *American Mathematical Monthly*, 1937, 44: 464–466.

The author considers a function $f(x)$ which satisfies the following conditions: (1) the equation $f(x) = 0$ has a single real root within an interval $0 < a \leq x \leq b$, which root is irrational; (2) the function $f(x)$ is single-valued and continuous in this interval; (3) the derivatives $f'(x)$ and $f''(x)$ are continuous and have no real zero in this interval. For such a function, the following theorem holds true: If Newton's method is applied a sufficient number of times to such a function, the value of the root of $f(x) = 0$ may be found to any desired degree of accuracy.

Conkwright, N. B. "Horner's Method and the Algorithm for the Extraction of Square Roots and Cube Roots." The Mathematics Teacher, 1944, 37: 31–32.

Corey, S. A. "Method of Solving Numerical Equations." *American Mathematical Monthly*, 1933, 40: 163–164.

A variation of the customary iterative

methods, applicable to transcendental equations as well as polynomial equations.

Cronvich, L. L. "On the Graeffe Method of Solution of Equations." *American Mathematical Monthly*, 1939, 46: 185–190.

Recommends that Graeffe's method be used for the solution of equations containing irrational roots and imaginary roots which are desired only approximately; a further discussion and extension of the eleven rules given by Hutchinson (q.v.).

Didden, C. A. "Practical Solution for Equations of Higher Powers." *Civil Engineering*, 1943, 13: 327.

Douglas, W. A. "Higher Degree Equations Solved by Recurrent Methods." *Product Engineering*, 1943, 14: 451–452; 672–673.

Faber, G. "Über die Newtonsche Näherungsformel." *Journal für Reine und Angewandte Mathematik*, 1910, 138: 1–21.

Frame, J. S. "A Variation of Newton's Method." *American Mathematical Monthly*, 1944, 51: 36–38.

A method which uses the approximation formula

$$X_2 = x_1 + \frac{-y_1}{y_1' + (y_1''/2)(-y_1/y_1')},$$

instead of the usual approximation formula of Newton's method.

———. "Remarks on a Variation of Newton's Method." *American Mathematical Monthly*, 1945, 52: 212–214.

Another iterative method, more general than Bailey's method.

Gibbins, N. M. "Application of the Newton Approximation to the mth Root of N." *Mathematical Gazette*, 1944, vol. 28, Mathematical Notes, No. 1781.

Hatch, Hollis D. "Horner's Method—Shortened." *School Science and Mathematics*, 1936, 36: 1007–1008.

When using Horner's method, the depressed equation is of the type ending in . . . $bx^2 + cx + k = 0$; the next approximation is found by taking $cx + k = 0$, or $x = -k/c$. The author suggests that the following approximation be used instead:

$$\frac{1}{x} = \frac{b}{c} - \frac{c}{k}.$$

By so doing the most tiresome last two or three steps of the usual procedure are avoided without loss of accuracy when finding a root, for example, to six decimal places.

Hausmann, B. A. "Graeffe's Method and Complex Roots." *American Mathematical Monthly*, 1936, 43: 225–229.

Introduces a modification of Graeffe's method, extending it to the case where there are more than two pairs of complex roots in a given equation, and simplifying the procedure in such cases.

Heinrich, H. "Das Hornersche Schema als Hilfsmittel bei der Untersuchung ganzer rationaler Funktionen." *Zeitschrift für Mathematischen und Naturwissenschaftlichen Unterricht*, 1933, 64: 331–336.

Hutchinson, C. A. 'On Graeffe's Method for the Numerical Solution of Algebraic Equations." *American Mathematical Monthly*, 1935, 42: 149–161.

An excellent description of Graeffe's method with illustrative examples worked out.

Hüttig, F. "Über Näherungsauflösung von Gleichungen." *Zeitschrift für Mathematischen und Naturwissenschaftlichen Unterricht*, 1928, 59: 436 ff.

Jeffreys, H. "Numerical Solution of Algebraic Equations. *Nature*, April 1927, 119: 565.

Lowry, H. V. "Approximation to the Roots of an Equation." *Mathematical Gazette*, 1945, 29: 225 ff., Mathematical Notes, No. 1848.

Mitchell, Merle. "An Interesting Algorithm." *The Pentagon*, 1943, 2: 69–73.

A short-cut method for finding the values of a polynomial and the numerical solution of equations, using a modified method of synthetic division.

"On Root Approximation." *National Mathematics Magazine*, 1943, 17: 312–314.

A method of finding the approximate value of the root of a polynomial equation by successive substitutions, using synthetic division, rather than by diminishing the roots of successively derived equations, as in Horner's method.

Pawley, Myron G. "New Criteria for Accuracy in Approximating Real Roots by the Newton-Raphson Method." *National Mathematics Magazine*, 1940, 15: 111–120.

The expression commonly given in the literature for the inherent error involved in the Newton-Raphson method is shown to be incorrect. New criteria are derived which may be safely used, and which may likewise be applied to extensions of the Newton-Raphson method involving derivatives of higher order.

Shain, J. "Method of Cascades." *American Mathematical Monthly*, 1937, 44: 24–29.

A method invented by Michel Rolle (1690), which consists essentially of first finding two numbers between which the root of a polynomial equation lies, and then converging upon the root by the use of cascades. This means the process of taking successive derivatives of the given equation, which is first suitably transformed so that all its real roots are positive. A cascade of an equation is its derivative, and the principle involved is that the roots of an equation are separated by the roots of its cascade.

Steen, F. H. "A Method for the Solution of Polynomial Equations." *American Mathematical Monthly*, 1937, 44: 637–644.

A method for finding a fractional approximation to a real root of a rational integral equation with numerical coefficients. The method described usually requires less computation than is required for Horner's method or for the customary extraction of square roots or cube roots, and is better adapted to

the use of computing machines. It is especially valuable where a high degree of accuracy in the approximation is desired.

Waidelich, D. L. "Numerical Solution of Equations." *Electrical Engineering*, 1941, 60: 480–482.

Wall, H. S. "A Modification of Newton's Method." *American Mathematical Monthly*, 1948, 55: 90–94.

Instead of using the conventional formula of Newton's method, viz.:

$$x_{p+1} = x_p - \frac{f(x_p)}{f'(x_p)}, \qquad (p = 0, 1, 2, \cdots),$$

it is suggested that the following formula might be used:

$$x_{p+1} = x_p - \frac{f(x_p)}{f'(x_p) - \dfrac{f(x_p)f''(x_p)}{2f'(x_p)}}, \qquad (p = 0, 1, 2, \cdots).$$

2. On the Location of Roots

Ballou, D. H. "On the Location of the Roots of Real Polynomial Equations when Two Roots are Equal." *American Mathematical Monthly*, 1939, 46: 209–212.

Buchner, P. "Der Sturmsche Satz in graphischer Darstellung." *Zeitschrift für Mathematischen und Naturwissenschaftlichen Unterricht*, 1927, 58: 61 ff.

Corliss, J. J. "Upper Limits to the Real Roots of a Real Algebraic Equation." *American Mathematical Monthly*, 1939, 46: 334–338.

Kennedy, E. C. "Concerning Nearly-equal Roots." *American Mathematical Monthly*, 1941, 48: 42–43.

———. "Concerning Upper and Lower Bounds of the Roots of a Real Algebraic Equation." *National Mathematics Magazine*, 1939, 14: 76–80.

———. "Root Isolation through Curve Analysis." *National Mathematics Magazine*, 1940, 14: 373–378.

Moorman, R. H. "A Lower Limit of the Roots of a Rational Integral Equation." *Scripta Mathematica*, 1945, 11: 379–380.

Myers, R. J. "Upper Limit for the Roots of an Equation." *American Mathematical Monthly*, 1933, 40: 35–36.

Westerfield, E. C. "New Bounds for the Roots of an Algebraic Equation." *American Mathematical Monthly*, 1931, 38: 30–35.

Wilson, N. R. "Separating the Roots of Polynomial Equations; a Modification of Sturm's Theorem." *Royal Society of Canada, Transactions*, 1935, Ser. 3, 29 sec. 3: 69–76.

3. Miscellaneous Methods of Solving Cubic and Quartic Equations

Breidenbach, W. "Zur Berechnung der Wurzeln kubischer Gleichungen in der Zahlenebene." *Zeitschrift für Mathematischen und Naturwissenschaftlichen Unterricht*, 1926, 57: 70 ff.

Davis, W. R. "Graph Solves Cubic Equation when Cardan's Formula Fails." *Civil Engineering*, 1948, 18: 100.

Fettis, H. E. "On Various Methods of Solving Cubic Equations." *National Mathemaics Magazine*, 1942, 17: 117–130.

Grant, J. D. "Graphical Solution of the Quartic." *American Mathematical Monthly*, 1933, 40: 31–32.

Hacke, J. E. "A Simple Solution of the General Quartic." *American Mathematical Monthly*, 1941, 48: 327–328.

Llewellyn, F. T. "Solving Cubic and Quartic Equations; Method for Use in Every-day Structural Design." *Engineering News Record*, 1930, 105: 254–256.

Lorey, W. "On Dieffenbach's Method for the Solution of Biquadratics." *National Mathematics Magazine*, 1937, 11: 217–220.

McMahon, F. J. "New Method for the Solution of Cubic Equations." The Mathematics Teacher, 1947, 40: 33–35.

Miller, G. A. "Solution of the Cubic Equation." *Science*, 1944, 100: 333–334.

Nogrady, H. A. "New Method for the Solution of Cubic Equations." *American Mathematical Monthly*, 1937, 44: 36–38.

Pettit, John T. "A Speedy Solution of the Cubic." *Mathematics Magazine*, 1947, 21: 94–98.

Plummer, H. C. "Numerical Solution of a Type of Equation." *Philosophical Magazine*, 1941, s7, 32: 505–512.

Pöschl, Theodore. "Ein Verfahren zur angenäherten Auflösung der kubischen und biquadratischen Gleichungen." *Zeitschrift für Mathematischen und Naturwissenschaftlichen Unterricht*, 1934, 65: 329–336.

Risselman, W. C. "On the Solution of Cubic Equations." *American Mathematical Monthly*, 1932, 39: 229–230.

Ritter, A. "Graphical Method for Cubic Equations." *School Science and Mathematics*, 1915, 15: 804–805.

Rocard, Y. "Méthode pratique pour la résolution de l'équation du quatrième degrè." *Revue Scientifique*, 1939, 77: 4–7.

Running, T. R. "Graphical Solutions of Cubic, Quartic and Quintic." *American Mathematical Monthly*, 1943, 50: 170–173.

Sah, A. Pen-Tung. "A Uniform Method of Solving Cubics and Quartics." *American Mathematical Monthly*, 1945, 52: 202–206.

Schucker, M. G. "Solution of Cubic Equations by Straight Line Graphs." *School Science and Mathematics*, 1920, 20: 818–820.

Widder, W. "Bestimmung reeller Wurzeln bei Gleichungen dritten Grades." *Zeitschrift für Mathematischen und Naturwissenschaftlichen Unterricht*, 1930, 61: 116 ff.

Yates, Robert C. "A Mechanical Solution of the Cubic." The Mathematics Teacher, 1939, 32: 215.

BOOK SECTION

Edited by J. STIPANOWICH
Western Illinois State College, Macomb, Illinois

THIS section presents the latest books which have been received for review in THE MATHEMATICS TEACHER. Reviews of many of these books will appear in the monthly issues. Members of the Council are invited to send us further comments or corrections of errors relating to any of the books mentioned. In addition, a free loan service has been arranged whereby any member may borrow any of the books listed for a period not to exceed two weeks. Requests should be addressed to THE MATHEMATICS TEACHER, 212 Lunt Building, Northwestern University, Evanston, Illinois.

BOOKS RECEIVED

HIGH SCHOOL

General Mathematics

General Mathematics by Virgil S. Mallory, State Teachers College, Montclair, New Jersey; and Kenneth C. Skeen, Union High School, Taft, California. Cloth, vii+478 pages, 1951. Benj. H. Sanborn and Co., 221 E. 20th Street, Chicago.

Algebra

First Algebra by Virgil S. Mallory, State Teachers College, Montclair, New Jersey. Cloth, vii+443 pages, 1950. Benj. H. Sanborn and Company, 221 E. 20th Street, Chicago.

COLLEGE

Analytical Geometry

Analytical Geometry by R. Walker, Mathematical Master, Stowe School, Sometime Scholar of King's College, Cambridge. Cloth, 248 pages, 1950. Edward Arnold and Company, London; Longmans, Green and Co., Inc., 55 Fifth Avenue, New York. $2.25.

Calculus

Operational Calculus by Balth. Van Der Pol and H. Bremmer. Cloth, xii+415 pages, 1950. Cambridge University Press, American Branch, 51 Madison Avenue, New York 10, N. Y. $10.00.

Advanced Mathematics

Elementary Theory of Equations by Samuel Borofsky, Brooklyn College. Cloth, x+302 pages, 1950. Macmillan Company, 60 Fifth Avenue, New York 11, N. Y. $4.25.

TEACHING OF MATHEMATICS

Space and Geometry by Aaron Bakst, School of Education, New York University. Paper, iii+77 pages, 1950. New York University Bookstore, Washington Square, New York 3, N. Y. $0.90.

REVIEWS

Arithmetic in Agriculture. Theodore H. Fenske, Richard M. Drake, and Allen W. Edson. St. Paul, Webb Publishing Co., 1950. x+243 pp., $1.60.

The trend toward making arithmetic practical has been followed in this newest publication of arithmetic for the farm student. Throughout the text the authors have consistently kept in mind the interest of the pupil and even in the review of fundamental processes, problems pertain to farm situations.

This compact text can easily be used as a source book by the student in later years as it contains many useful tables and a complete index. The authors suggest the use of supplementary material to enrich the course whenever there is a need or opportunity to introduce it. Illustrations and graphs add to the attractiveness of the book. The content of the chapters entitled "The Use of Measures on the Farm," "Farm Credit and the Wise Use of Money," "Crops, Soils and Fertilizers," "The Dairy Herd," "General Livestock and Poultry," "Farm Building and Engineering," and "Farm Economics and Management," should help the student apply basic arithmetic to farm problems.—HUMPHREY C. JACKSON, John D. Pierce Junior High School, Grosse Pointe, Michigan.

Everyday General Mathematics. William Betz, A. Brown Miller, F. Brooks Miller, Elizabeth B. Mitchell, and H. Carlisle Taylor. New York, Ginn and Co., 1949. xii+436 pp., $2.40.

This book incorporates the advances that have been made in general mathematics in recent years, but at the same time retains the best features of the old methods. The book has a wealth of problem material, drill material, diagnostic and remedial features, provisions for measuring achievement, and provisions for individual differences. It includes material in which emphasis is placed on understanding, interpretation, and use.

Sixty percent of the material utilizes the basic concepts of numbers as used in actual life situations. For example: 7.5 tons of coal at $18.00 per ton; shop drawings which show eight holes to be drilled; planning to make six aprons; planning the family budget and instalment buying. Less than twenty percent of the material presented is called "Geometry" and the authors have succeeded in presenting this section as practical problems in mensuration. Slightly more than twenty percent of the material is called "Alge-

bra" and is presented in a very simple and elementary manner.

The teacher will find the book suitable for classes of average ability and also easily adjusted for the retarded classes. The text should definitely be considered when selecting a text for general mathematics.—KENNETH ALLEE, Hartford City High School, Hartford City, Indiana.

A Second Course in Algebra (Second revision). N. J. Lennes and J. W. Maucker. New York, The Macmillan Co., 1950. xv+522 pp., $2.48.

The basic philosophy of Lennes and Maucker centers in the idea that the average student can learn more by doing a simple example through his own efforts than by struggling with a difficult example which requires continuous and detailed explanations by the book and the teacher. The emphasis is always on *why*, not *how*.

The text is amply supplied with sight work, exercises and problems, as well as Chapter Reviews, Cumulative Reviews and Special Practices. There is no lack of examples in this book. Exercises, problems, and supplementary material are labeled (A), (B), (C)—thus grading the work in three steps from easy through difficult, the (C) being included primarily for the neglected bright students we are talking so much about and for whom we are doing so little.

The type is excellent; the illustrations are excellent; the language is simple and direct. Sample examples are plentiful and effectively enclosed in rectangles. Chapters twenty through twenty-three take up permutations and combinations, probability, determinants, and a historical sketch; thus providing additional material for the abler students.

I enthusiastically commend this book to the attention of all second-year algebra teachers.—DOROTHY S. WHEELER, Bulkeley High School, Hartford, Conn.

Commercial Algebra (Third Edition). Thomas M. Simpson, Zareh M. Pirenian, and Bolling H. Crenshaw. New York, Prentice-Hall, Inc., 1950. xi+173 pp., $2.50.

This book is designed to meet the needs of those who wish (1) preparation for a course in compound interest and annuities, or (2) a course in college algebra of a more practical nature than the traditional courses.

It differs from the usual college algebra text in that a chapter on elementary statistics is included, and many of the applications involve interest, discount, partial payment, equation of payments, equation of value, equation of accounts, and other commercial problems. Among topics usually found in college algebra texts which are absent from this book are variation, synthetic division, remainder and factor theorems, permutations, combinations, probability, mathematical induction, and De Moivre's Theorem.

The teacher of a traditional course in college algebra would object to some of the omissions. For the purposes stated, however, the book seems to be adequate, and well organized. Features particularly appealing to the reviewer are: (1) the explanation of and exercises in "translation" from word language to algebraic language, and (2) the wealth of illustrative examples and review exercises.

Instructors of courses having the purposes mentioned above should examine this text carefully.—CLEON C. RICHTMEYER, Central Michigan College, Mount Pleasant, Michigan.

Primer of College Mathematics. John F. Randolph. New York, The Macmillan Company, 1950. xii+544 pp., $4.75.

This is another attempt to unify the material usually found in courses in college algebra, trigonometry, and analytic geometry. There is also an introductory chapter on the calculus. The unifying idea is the notion of functions and their representations.

The chapter titles gives an idea of the arrangement of material. They are: (1) "Arithmetic, Algebra, Probability," (2) "Functions and Rectangular Coordinate Systems," (3) "Equations, Inequalities, Identities, Determinants," (4) "The Binomial Theorem and Applications," (5) "Logarithms," (6) "Introduction to Calculus," (7) "Trigonometry," (8) "Solution of Triangles," (9) "Graphical Analysis," (10) "Theory of Equations and Identities," (11) "Solid Geometry."

The author points out that he has presented the work on solutions of triangles in Chapter 8 and the standard material on conic sections in Chapter 9 in such a way that they may be easily omitted. Why one should wish to do so is not quite clear.

There seems to be an adequate supply of exercises with answers provided for about half. A brief set of tables and an index are included.—FRANK C. GENTRY, Arizona State College, Tempe, Arizona.

Analytic Geometry and Calculus. Harold J. Gay and Raymond K. Morley. New York, McGraw-Hill Book Company, 1950. vii+524 pp., $5.00.

This is a text which is designed to integrate the teaching of analytic geometry and calculus. Most of the proof seemed to be presented in a form somewhat comparable to methods found in traditionally organized textbooks in the separate subjects.

The attempt to integrate the material, however, makes for a much more meaningful development of concepts, and is more likely to insure the establishment of relationships between the various fields of mathematics included.

The textbook is carefully written so as to make it possible for students to follow, in a very carefully organized step by step manner, the most appropriate procedures for the solution of problems. If carefully used by instructors, this book should eliminate a great deal of repetition in the development of new mathematical concepts in both analytic geometry, as well as in the differential and integral calculus.—HARRY C. JOHNSON, University of Minnesota, Duluth Branch, Duluth, Minnesota.

LETTERS

Questions on Per Cent

Consideration of two questions on per cent in the Sueltz paper in the January 1951 MATHEMATICS TEACHER leads to further questions about per cent. He asked college freshmen, considered at least average in intelligence: How is one fourth of one percent written as a decimal? A large orange contains 150% more juice than a small one. This is how many times as much juice?

Correct answers selected: 1st question, 27%; 2nd, 16%.

From this should we conclude that per cent ought to be taught better at some level; or that it ought not to be taught at all?

Is this awkward double standard—both hundredths and per cents for expressing the same parts—due merely to business men's lack of complete faith in the decimal system, and failure to apply it in this situation?

For those who believe that this is so:

Is education a force which should stand up against business when business is foolish and wasteful in such a practice? Or should education teach merely what business wants taught? Recalling how many generations it took business to accept the decimal system originally, should education now resist this clumsy practice, or consider resistance helpless and say nothing?

Suppose that we decided to fight, and to show every pupil, by grade IX, that per cent is a useless, burdensome, expensive, extra terminology. Could the resulting effect on the everyday use of percent, after say 20 years, give a sound test of the effectiveness of our education?

HARRY C. BARBER
Newton Center, Massachusetts

Answer to Harry C. Barber on Per cent

Questions:

First: Should *per cent* be taught better at some level?

Second: Should *per cent* be taught at all when decimal fractions serve the same purpose?

Answers:

The answer to the second question involves, among other things, the elements of human nature and a basic ethic. Can the schools in one generation or even in 100 years change a mode of expression and thinking which society has adopted? If this could be done, should it be done by the schools? Historically it seems easier for the population to adopt new ideas and procedures than it is to drop old patterns. This seems particularly true when the older patterns are identified with a functional use. It chagrins many mathematics teachers to note the use of fractions (common and decimal) currently used in market quotations: 151⅜ and 1 11/32 (stock); 109.16 (meaning 109 and 16/32, bond); .3229 (cotton); and .9016 and .73¼ (hides). Apparently business continues in use and invents or adopts practices that serve its needs but does so without due regard to best possible practice in terms of all of the elements involved. Can the schools change this? Stevin and later Erastus Root and others tried to drop common fractions but the public did not follow their lead. Assuming that the proposal to drop *per cent* has merit and that this could be achieved, should the schools set out deliberately to achieve this goal? That is a serious question. Should the schools lead society or merely interpret it? My own impression is that, in a democracy, our larger emphasis should be devoted to the role of interpretation and to provide ideas for bettering society but should not attempt to remake society by fiat. Without arguing the merits of *per cent* in comparison with *decimals*, I doubt that we should attempt to make this change in the public schools.

The first question. Assume that *per cent* will continue to be used in our society. Recognize many per cents used in business and industry are not the simple ones to which the elementary school restricts its teaching. Then, to provide an intelligent citizenry, *per cent* ought to be taught at an age level at which the more complex numbers and situations can be made meaningful and intelligible to pupils. This does not mean that percentage should be removed from the elementary school. It argues for the extension of basic learning at a more advanced level. *In the high school.* The extension of arithmetic topics into the high school is desirable. Where can we find the time? Would mathematics teachers and curriculum workers agree that, for a large part of our population, the extension of arithmetic is more valuable and important than certain parts of algebra and perhaps half of the typical one-year course in plane geometry? If *per cent* were dropped and *decimal* used exclusively we would still be faced with the problem of meeting the needs of society by instruction beyond grade eight.

To Mathematics Teachers:

Consider Mr. Barber's proposition very seriously. Assuming that it could be done, do you believe that the best interests of our country would be served by dropping the terminology and framework of *per cent* and basing all of this work on the idea and notation of *decimal?* Shall we argue the proposition even though you may consider this academic?

BEN A. SUELTZ
Cortland, New York

Solving Equations by the Factoring Method

In an article in the March 1948 issue of THE MATHEMATICS TEACHER, William R. Ransom recommends that the "factoring method" of solving equations be dropped in favor of a "division, two-case method." The fault he finds with the "factoring method" is that most students cannot be taught to understand or to use properly the theorem that "If a product is zero, at least one of the factors is zero."

Teachers and textbook writers who use this theorem in explaining the "factoring method" should not expect good results. The reverse theorem is the proper one to use: "A product is zero if any one of its factors is zero." Then, if the pupil is aware of what he is seeking (and a good feature of the "factoring method" is that it requires this understanding of the definition of a root), it is easy for him to see the value of having his equation in the form of a product of factors equal to zero. He understands why setting *each* factor to zero leads to an answer, and he is not worried by such doubts as "If I know only that *at least* one of these is zero, what right have I to set both of them to zero?"

Later on, if it is desired to show that this method gives all the answers, the theorem can be elaborated to include "and only if." Mr. Ransom's "two-case method" has merit (although it cannot easily be used for equations of higher degree because of the need of sub-cases), but let us not discard the "factoring method"!

KENNETH W. WEGNER
Northfield, Minnesota

Covers for THE MATHEMATICS TEACHER

Has anyone ever suggested the possibility of covers for combining all the issues of a single volume into a single book? If this could be done feasibly, I for one would be very happy, as this would enable me to keep my file of THE MATHEMATICS TEACHER all together and in one place where it belongs.

Also has the idea of membership cards ever been broached? This would seem to me to bring a certain definiteness to the membership such as is brought about by other professional groups who use membership cards.

FRANK J. HARDMAN
Baltimore, Maryland

EDITOR'S NOTE: How many readers would purchase covers, if they could be made available? We will welcome your postcard reply.

A Problem Department

I wish to take this opportunity to suggest the addition of a Problem Department in THE MATHEMATICS TEACHER. Problems of varying difficulty could be offered in one issue and solutions (submitted by readers) in a subsequent issue.

H. SIDOFSKY
New York, N. Y.

MEETINGS OF AFFILIATED GROUPS

Colorado

Colorado Council of Teachers of Mathematics: Spring meeting in late April or early May

Mathematics Section, Eastern Division, Colorado Education Association: April 20–21, Colorado State College of Education, Greeley

Florida

Florida Council of Teachers of Mathematics: March 31, Tampa; August, Gainesville

Georgia

Georgia Council of Teachers of Mathematics: March 23–24, Atlanta; two days in May, Milledgeville; eleven meetings by districts in October

Illinois

Men's Mathematics Club (Chicago and Metropolitan Area): March 16, Chicago Bar Association; April 20 and May 18, YMCA Hotel

Women's Mathematics Club of Chicago and Vicinity: March 16, Chicago Bar Association; May 5, Museum of Science and Industry

Indiana

Indiana Council of Teachers of Mathematics: Annual Meeting, April 28, Indiana University, Bloomington

New England

Association of Teachers of Mathematics of New England: May 12; August 23–30, Summer Institute at Connecticut College, New London

New Jersey

Association of Mathematics Teachers of New Jersey: May, Rutgers Univ., New Brunswick

Ontario

Ontario Association of Teachers of Mathematics and Physics: March 27–28, Toronto

Tennessee

Mathematics Section, Eastern Tennessee Education Association: October 26, University of Tennessee

Texas

Mathematics Section, Texas State Teachers Association: Eleven district meetings in the spring

Virginia

Richmond Chapter, National Council of Teachers of Mathematics: April 16, W.R.N.L. Theatre

Wisconsin

Wisconsin Mathematics Council: Joint meeting with the Wisconsin Section, Mathematical Association of America, May 12, Carroll College, Waukesha

AIDS TO TEACHING

Edited by

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BOOKLETS

B.56—Personal Money Management

Local representative of American Bankers Assoc., 12 East 36th Street, New York 16, New York.

Booklet; 36 pp.; 7″×9¾″; Single copies—$.25.

Description: The booklet is composed of eleven brief essays on methods of controlling expenditures economically and on means of determining the suitability of one's present spending habits. Individual chapters are devoted to the factors which consume the largest portions of one's income—savings, shelter, household operating expenses, food, clothing, and advancement. The final chapter of the booklet is especially good; it is entitled, "Are You as Old Financially as in Years?" The author classifies all people in three somewhat artificial, but nevertheless useful, categories according to savings—minimum, middling, and master. A clever graph serves to illustrate the "financial life-line" of people within these classes so that one may plot his own economic condition and determine his monetary age. A second excellent feature of the booklet is an enclosed table, "Suggested Distribution of Income After Income Taxes." It is an up-to-date revision of a table which is in the booklet. Budgets are suggested for families of two, three, and four members whose incomes range between $150 and $1,000 per month.

The booklet is printed upon light brown paper in dark brown print. The format is excellent in every respect. Each chapter is followed by a helpful bibliography for those who care to pursue the subjects further; the chapters are well-arranged and presented in terms understandable to the layman.

Appraisal: The topics presented are certainly consistent with the general objectives of socialized mathematics courses. Increased ability in money management is a skill vital to every wage earner.

Each of the essays has been written by a person well qualified to expound upon theories in his particular field; the essays are therefore authoritative. They are probably most useful on the senior high school level for the vocabulary is difficult for younger students.

The teacher may well utilize the booklet in directing brief projects which require solutions to problems involving present and future financial status of the pupils. Such work may well contribute toward the students' formation of vocational plans.

It would indeed be advantageous to supply each student with an individual copy, but where the school budget does not permit, the teacher can reproduce the context in class and make the single copy available for student examination. The booklet is certainly worth twenty-five cents. (Reviewed by Bernard Singer, Hyannis, Mass.)

CHARTS

C.21—Cross Section Chart

Central Scientific Co., 1700 Irving Park Road, Chicago 13, Illinois.

Chart; 40″×40″; $7.50.

Description: This heavy fabric chart is coated with a black rubberized substance; forty horizontal lines and a like number of

vertical ones form the graph. The heavy yellow lines are drawn at one-inch intervals and every fifth one is particularly heavy. Contrast between the yellow and black colors offers excellent visibility. Chalk may be used on the rubberized surface.

Wooden rollers at top and bottom of the chart cause it to hang correctly when in use but its flexibility is a definite liability for it must be placed directly against a wall to allow drawing thereupon. The surface of the chart may be easily pricked and will not withstand many years of use.

Appraisal: A graph board such as this is a poor substitute for the more common type which has lines impressed upon a rigid blackboard surface. It does have two good points: the visibility of the graph lines is excellent and the chart occupies a minimum of space when not in use. (Reviewed by Bernard Singer, Hyannis, Mass.)

C.22—Graph Chart

J. L. Hammett Company, Kendall Square, Cambridge 42, Mass.

Chart; 36″×36″; $7.00.

Description: The chart is painted in white upon blackboard cloth reinforced by canvas; it has common stick rollers at top and bottom. The graph itself is painted in one-inch squares.

Similar charts are available with a spring roller mounting or painted upon a Hyloplate board. The price with the spring roller is $9.50, with the Hyloplate, $12.00.

Each model has its advantages. The lowest price may make the first model desirable; the desirability of hanging from the blackboard frame without a storage problem may make the spring roller chart preferable; rigidity of the Hyloplate board and consequent simplification of drawing may lead to a choice of this chart. Selection must be made in terms of individual preferences.

Appraisal: There is little need to justify the use of a rectangular coordinate chart by the mathematics teacher. Each of the models above may satisfy a real need in classrooms which do not have a permanently-engraved graph on the wall blackboard. The price of these graphs (and others) is somewhat expensive due to recent price increases. (Reviewed by Bernard Singer, Hyannis, Mass.)

C.23—Mathematics Calendar

Miss Laura E. Christman, 1217 Elmdale Ave., Chicago, Ill.

Chart; Ten charts; 11″×17″; Copyright 1935; 50¢.

Description: This calendar is a series of ten charts and a cover chart which outlines briefly some of the important events in the development of mathematics from 4000 B.C. to 1917 A.D. Each chart covering a certain period of history has a series of drawings with a brief listing of the facts discovered during that period. Each chart presents the mathematics according to the country in which it developed as follows: Babylonia and Egypt 4000–3000 B.C.; Egypt 2000–1700 B.C.; Greece 640–300 B.C.; Alexandria 287 B.C–300 A.D.; Persia 500–1200 A.D.; France and Italy 1300–1600 A.D.; England and Germany 1666–1754; The World 1754–1917. The charts are printed by photographic reproduction on durable paper.

Appraisal: These charts will furnish the mathematics teacher with material for display in the classroom to tell the story of mathematics. It omits specific dates for the discovery of facts which may be desirable. Instead, it shows general trends and the contribution of the different sections of the world to the development of mathematics. The drawings are appropriate but often lack interest since they are similar to textbook drawings. It is unfortunate that the facts on each chart are stated in such brief outline form that they lack meaning for the high school student. The role of mathematics needs to be presented in our mathematics classes. More materials and better materials like these

charts are needed to help the mathematics teacher present the story of mathematics. Many materials of this type can be made by high school students if they are stimulated by the teacher. These charts will give the teacher ideas as to possible ways of illustrating historical events.

FILMS

F.58—How to Find the Answer

Coronet Films, Coronet Building, Chicago, Ill.

Film; 16 mm. sound; 1 reel; black and white.

Description: This film attempts to show problem solving in a natural setting by following the processes used by three youngsters of junior high school age in solving problems directly related to their experiences. The problems—the purchase of a wood lathe on an installment plan, measurement of the living room curtains in determining how much material should be purchased for new curtains, and determining the number of boxes of floor tile needed to cover the kitchen floor—all emphasize the need for having the necessary data or information before the solution of a problem can be obtained. Solutions of each problem are carefully discussed and reviewed with necessary information stressed. A summary outline highlights the similarity of important points in the solution of each problem.

Appraisal: Filming, sound, and presentation of material is good. The film effectively illustrates the need of compiling data before the solution of the problem can be effected. Closeups of computational work shown are presented in neat form and easily understood. However, the film neglects to point out that with the given data all of the problems might have been solved using different methods of computation. The film would probably be most effective as a summary on a unit for problem solving, although it could be used to introduce such a unit with careful planning and preparation. Contents of the film are applicable to eighth and ninth grade levels and in some instances to advanced seventh grade mathematics. The problems presented are of a maturity level suited to the junior high school and present a variety that adds to the interest of the film. The actors and their acting are natural enough to be effective. (Reviewed by Nick Lovdjieff, University of Minnesota High School, Minneapolis, Minn.)

FILMSTRIPS

FS. 78—Length

FS. 79—Mass

FS. 80—Time

Cultural Films Incorporated, 62 William St., New York 5, N. Y.

35 mm. filmstrip; black and white; silent; 47 frames; $3.00.

Description of FS. 78: This is Part I of the Length, Mass and Time series. The first 10 frames are used for introduction showing how measurement enters our lives. The next eight frames give the historical development of different units of length, historical as well as present. The next 23 frames compare the metric and English systems, define the meter, and show the micrometer, vernier caliper, and Johannsen gauge blocks. The last six frames deal with distances between the earth and other planets and mention distance measured in light years.

Appraisal of FS. 78: Except for one or two frames the photography is satisfactory. A wealth of material is mentioned but there is a lack of continuity in the frames.

The measurements in the English and metric systems could be shown as well by the use of the yardstick and meter stick in the classroom. However, the derivation of the meter and the picture of the meter bar in Washington are interesting.

If the material in the filmstrip coincides with that of the course it could be used by showing a few frames at a time. But I would like to emphasize that it would

take much preparation and explanation to the class by the teacher.

Description of FS. 79: This is Part II of the Length, Mass, and Time series. The first five frames show mass and/or matter. The next 7 frames explain weight, show Newton and the apple, gravitational pull of planets, comparative weights of the planets, and that far enough away from planets in space, man would have no weight, only mass. The next three frames deal with differing weights at different points on the earth, and also two ways of measuring them. The next frame shows two trucks, one loaded and the other empty, and speaks of "heavier" mass making the loaded truck harder to stop and start. The next 8 frames deal with the metric system of measure and compare large and small bodies of matter. The last 18 frames give historical notes in the even arm balance and old weights which were used. Then pictures of some of the weighing machines used today are shown.

Appraisal of FS. 79: The photography is satisfactory with the exception of some of the details on modern weighing machines. In this filmstrip there is also a lack of continuity of presentation. The frame with the two trucks seems out of place since it is apparently concerned with acceleration rather than mass and weight. Also the use of the term "heavier" referring to mass seems inappropriate in a filmstrip attempting to differentiate between weight and mass. The history of the beam balance is interesting, but the pictures of present day weighing machines lose interest because of lack of explanation.

Description of FS. 80: This is Part III of the Length, Mass, and Time series. The first nine frames are taken up with definitions of time as sequence of events, picture of the calendar, and the earth rotating around the sun, as well as an explanation of leap year. The next four frames are used to show the Roman gods from whom the months derive their names. The following three frames show how the earth "spins around itself" and we have day. The next 17 frames show ancient timepieces, and the last 12 frames explain meridians and the change in time due to the earth's rotation. They also give the basic units of hours, minutes, and seconds.

Appraisal of FS. 80: The pictures of the Roman gods and the legends on these frames were not very clear. The filmstrip would be of interest to those studying antique timepieces, but otherwise it shows little that could not be given on the blackboard in the classroom. (Reviewed by Dale A. DuVall, Mathematics instructor, Boston University General College.)

INSTRUMENTS

I. 30—Chinese Abacus

Loy's Chinese Calculator, 1317 Rhode Island Ave., N.E., Washington 18, D. C.

Abacus; 2 sizes: $6'' \times 5\frac{1}{4}''$, $3.50; $10\frac{1}{4}'' \times 7\frac{1}{2}''$, $6.50; Booklets of instructions supplied with each model.

Description: Only the smaller model was seen to review. It has aluminum rods set in a cast aluminum frame, with beads of polished oak. There are five columns for computation, and the picture of the larger model shows eleven columns. Each, of course, has five beads below the cross bar and two above, since it is patterned on the Chinese abacus. The appearance of the small model is unusually sturdy, and looks as though it would last forever; the larger model is of wood, apparently from the picture, and was not tested. The booklet with the smaller model contains 17 pages of directions and 22 illustrations in order to teach computation using addition and subtraction only. The booklet for the other model is said to contain 60 illustrations, and so probably goes on to more difficult processes.

Appraisal: These are splendid devices to use and to enliven the appearance of mathematics classrooms. The booklet is well written, and uses the same method to teach computation that is used in Chinese

schools. It is probably too much to hope that the use will become as widespread as the descriptive leaflets describe, but the fun and the understanding of arithmetic processes which will be learned certainly justify its introduction into schools. Secondary schools and college classes can find wide applications of these devices in studying the foreign methods in arithmetic and in history of mathematics courses.

Even though the price seems high, the excellence of the construction seems to warrant it.

MODELS

M. 17—Plastic Globe

Herbach and Rademan, Inc., 522 Market Street, Philadelphia 6, Pa.

Globe; $9\frac{3}{4}''$ in diameter; plastic; catalog number TM1717; $.98.

Description: This transparent globe with a plane mounted in the center was originally used for training for the air force. It consists of a globe with markings indicating positions of guns and dangerous firing areas of the mounted plane. The globe is divided into 36 ten-degree divisions of longitude and 20-degree divisions of latitude. The small model plane is held in position by a metal rod which is removable.

Appraisal: This inexpensive war surplus article can be used in a variety of ways in the mathematics classroom. Because of its transparency it can be used to show map projections by mounting a light at the center. Other mathematical phases such as navigation, great circle routes, time zones, and spherical geometry can be illustrated by this globe. Although the plastic is advertised to be unusually strong and capable of withstanding considerable abuse, the writer has found it to be quite brittle. In any case, it is an excellent buy at 98¢.

M. 18—Plastic Figures

Vis-X Company, 1049 South Flower, Los Angeles, California.

Models; 5 plastic liquid measures; $10.00.

Description: The five-piece plastic set consists of a liquid-quart cylinder, a cone of the same base and altitude as the cylinder, a cube having inside measurements of ten centimeters, and a pyramid of the same base and altitude as the cube. The last piece is a band that fits on the top of the liquid quart enlarging it to a dry quart.

Appraisal: The attractive transparent plastic figures make it possible for the pupils to watch the process of filling the cylinder by pouring three conefuls of some colored material into the cylinder when developing the formula for the cone. Since the cylinder is a quart measure, and the cube is a liter measure, the relation of some of the basic measures of the English and Metric system may be shown. By using the band that transforms the liquid quart to a dry quart, we have a foundation unit for the peck and bushel. It is a practical, attractive, and valuable aid for any classroom. (Reviewed by Alice M. Phillipson, Mathematics Consultant, Los Angeles, Calif.)

SOURCES OF MATERIAL FOR LABORATORY WORK

SL. 21—Bulletin Board Wax

Lea A-V Service, Albert Lea, Minn.

Adhesive wax; $2.50 per box.

Description: This adhesive wax is a colorless, odorless material which can be used to mount three dimensional objects as well as flat materials on vertical surfaces such as bulletin boards, blackboards, walls, woodwork, glass and similar surfaces. According to the producer, when materials are properly mounted, they will remain in place indefinitely. The writer has found this to be true during the limited time he has used it. It may be used repeatedly, but loses adhesive strength if it collects foreign material such as dirt or dust. No tools, heat, or special applications are necessary. The warmth of the hand is sufficient to make the wax readily pliable and the amount desired can be easily pinched off. The wax is applied to

the back of the object or material to be mounted and is then pressed firmly against the mounting surface. The material can be dismounted by pulling the material from the display surface and any residue wax scraped away. On rough or porous surfaces a small amount of cleaning fluid will remove remaining wax. One box of wax is supposed to be a year's supply for one classroom.

Appraisal: This wax is a convenient material for making displays or posting pictures, charts, pupil work, etc. It eliminates the need for unsightly thumb tacks or adhesive tape as well as avoiding holes in woodwork or wall surfaces. It also makes it possible to mount materials on surfaces such as blackboards and windows. The only disadvantage of this material is that its adhesive strength is so great that there is a tendency for a residue to remain on the fingers as well as the display surface. However, this can readily be removed by scraping or by cleaning fluid. Since this material will greatly facilitate the valuable but time consuming process of displaying materials, it should be available in every classroom.

TRANSCRIPTIONS

TR. 1—Mr. Arithmetic

"Tutor that Never Tires," Inc; Bronxville 8, New York.

Drill recordings; six 10″ vinylite records in album, $10.00; Same in three sets of two records, $3.50 per set.

Description: The general form of these records is a series of number combinations read by a man's voice with a pause after each combination for the listener to answer. This is followed by the correct answer. There is some attempt to surround the drill with an atmosphere of a game, and an attempt to jolly the children into doing the work. The following is the material covered on the records:

1A: Adding 1–10. The combinations "2 and 2" through "2 and 8" are given, first in order and then mixed up. Then "3 and 3" through "3 and 7" the same way followed by "4 and 4", "4 and 5", "4 and 6" and "5 and 5." Then all combinations are scrambled.

1B: Adding Three Numbers. In the form of a "ladder game" three numbers are added; for example, "2 add 1 add 3." All sums are 10 or less, and zero combinations are introduced at the end.

2A: Adding 10–20. First, numbers that add to 11 (2 and 9, 9 and 2, 3 and 8, etc.) are given; then numbers that add to 12, then mixed-up sums of two numbers with totals from 11 to 20.

2B: Adding Three Numbers. The ladder game with four numbers: "2 add 3 add 3 add 2" . . . pause . . . "10." The answers are between 10 and 20.

3A: Subtraction 10–0. First, "10 take away 1" in order down to "10 take away 9" is followed by the same combinations in a mixed up sequence. Then, other combinations which subtract one digit from a larger one are given.

3B: Subtracting Three Numbers. The ladder game in this case is used in the form "10 minus 4 minus 2" . . . pause . . . "4."

4A: Subtraction 20—10. Problems in the form "20 take away 10," "19 take away 9," "13 take away 3" are followed by a mixed-up set similar to "19 take away 2," etc. All answers are between 10 and 19.

4B: Subtracting Three Numbers. Problems similar to "20 minus 6 minus 3."

5A: Multiplication to 36. This means to 6 times 6. It starts with "1 times 1," "2 times 2" through "6 times 6" then the same in a mixed-up sequence. Then general problems such as "2 times 4."

5B: Multiplication to 144. This means "7 times 7" through "12 times 12" and is similar to record 5A. However, combinations like "8 times 3" appear.

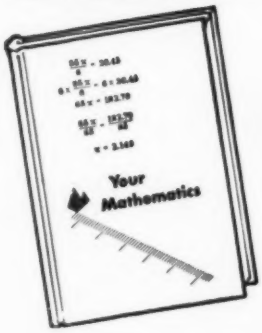
6A: Random Multiplication. Problems similar to "3 times 5," "6 times 1," etc., in a fairly well graded order are given.

6B: Random Multiplication, Harder. Again two digits are multiplied, but with more difficult combinations.

Appraisal: In general, the drill combinations are appropriately chosen and the use of recordings will add some novelty to the presentation. It may serve to relieve the teacher of some work and voice strain, or, as is evidently intended, make drill possible at home. Of course, a good teacher can invent examples better suited to the particular pupils and can supply a greater variety than the records can. More seriously, no motivation or advantages of using the phonograph record as a medium of teaching has been exploited. With a little imagination the producer could have used music, dramatization, odd and varying voices and other interesting devices. Instead a man's voice, and a somewhat bored one at that, reads all the combinations. He calls himself "Mr. 'Rithmetic" with a stressed mispronunciation. There is a disadvantage in not being able to adjust the speed of presentation and of answering to the pupils listening, or to the speed of a listener who is improving. Sometimes the time seems short for a beginner to answer. It must be admitted, however, that there is too little use of oral, mental arithmetic and these records may supply that need. The volume-level of the recordings is somewhat erratic, but not badly so. The physical appearance of the albums is most attractive and the unbreakable feature of the records will be greatly appreciated.

The novelty of such drill records will be appreciated by both teachers and pupils. These records are to be commended for a good attempt and should encourage further experimentation. For class use they are not necessary, but would be interesting. For home use, they may be very useful.

BUILDING MATHEMATICAL COMPETENCE



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